Krull, Gelfand-Kirillov and Filter Dimensions of Simple Affine Algebras

Vladimir Bavula

*

*Talks/lectbonn

Plan

1. Gelfand-Kirillov Dimension, Examples.

2. Filter Dimension.

3. Analog of the Inequality of Bernstein for Simple Affine Algebras.

4. Inequality between Krull, Gelfand-Kirillov and Filter Dimensions for Simple Affine Algebras. Applications to $\mathcal{D}$-modules.

1. Gelfand-Kirillov Dimension

module=left module

K is a field of char 0

N and **R** are sets of natural and real numbers

Definition. For a function $f : \mathbf{N} \to \mathbf{N}$, the real number or ∞ defined as

$\gamma(f) :=$ inf$\{r \in \mathbf{R} : f(i) \leq i^r$ for suff. large $i >> 0\}$

is called the **degree** (or **growth**) of f.

For functions $f, g : \mathbf{N} \to \mathbf{N}$:

$$\gamma(f + g) \leq \max\{\gamma(f), \gamma(g)\},$$

$$\gamma(fg) \leq \gamma(f) + \gamma(g),$$

$$\gamma(f \circ g) \leq \gamma(f)\gamma(g),$$

where $f \circ g$ is the composition of the functions.

Let A be an **affine** ($\equiv$ **finitely generated**) algebra with generators $x_1, \dots, x_n$. Then A is equipped with a standard finite dimensional **filtration**

$$A = \cup_{i \geq 0} A_i, \;\; A_0 = K,$$

$$A_1 = K + \sum_{i=1}^{n} K x_i, \;\; A_i := A_1^i, \; i \geq 2.$$

Let M be a finitely generated A-module and M_0 be a finite dimensional generating subspace of M, $M = AM_0$. The module M has a finite dimensional filtration

$$M = \cup_{i \geq 0} M_i, \;\; M_i = A_i M_0.$$

Definition (Gelfand-Kirillov, 1966). The **Gelfand-Kirillov** dimension of the A-module M:

$$\text{GK}(M) := \gamma(i \to \dim M_i).$$

The **Gelfand-Kirillov** dimension of the algebra A:

$$\text{GK}(A) := \gamma(i \to \dim A_i).$$

GK(M) and GK(A) do not depend on the choice of the filtrations.

Example. Let $P_n = K[X_1, \ldots, X_n]$ be the polynomial ring in n indeterminates.

$P_n = \cup_{i\geq 0} P_{n,i}$, $P_{n,0} = K$, $P_{n,1} = K + \sum_{i=1}^{n} KX_i$,

$P_{n,i} = \sum \{KX^\alpha \mid |\alpha| \leq i\}$, $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$,

$|\alpha| = \alpha_1 + \cdots + \alpha_n$.

- dim $P_{n,i} = \binom{n+i}{n}$

  $= (i+n)(i+n-1)\cdots(i+1)/n! = i^n/n! + \cdots$.

- GK(P_n) $= n$.

The n'th **Weyl algebra**

$$A_n = K < X_1, \ldots, X_n, \partial_1, \ldots, \partial_n >$$

defining relations:

$$\partial_i X_j - X_j \partial_i = \delta_{ij}, \text{ the Kronecker delta,}$$

$$X_i X_j - X_j X_i = \partial_i \partial_j - \partial_j \partial_i = 0, \; i, j = 1, \dots, n.$$

The algebra A_n is a simple Noetherian infinite dimensional algebra canonically isomorphic to the ring of differential operators with polynomial coefficients

$$A_n \simeq K[X_1, \dots, X_n, \partial/\partial X_1, \dots, \partial/\partial X_n],$$

$$X_i \leftrightarrow X_i, \;\; \partial_i \leftrightarrow \partial_i/\partial X_i, \;\; i = 1, \dots, n.$$

- $\{X^\alpha \partial^\beta\}$ is a K-basis of A_n.

- A filtration: $A_n = \cup_{i \geq 0} A_{n,i}$,

$$A_{n,i} = \sum \{K X^\alpha \partial^\beta, |\alpha| + |\beta| \leq i\},$$

$$\dim A_{n,i} = \binom{2n+i}{2n} = i^{2n}/(2n)! + \cdots.$$

- $\mathrm{GK}(A_n) = 2n$.

2. Filter Dimension

Lemma 1. *Let A be a simple affine inf. dim. algebra and let $M \neq 0$ be a f.g. A-module. Then* $\dim M = \infty$, *hence* $\text{GK}(M) \geq 1$.

Proof. The alg. A is simple, so the nonzero algebra homomorphism

$$A \to \text{Hom}_K(M, M), a \mapsto (m \mapsto am),$$

is injective, so $\infty = \dim A \leq \dim \text{Hom}_K(M, M)$ and $\dim M = \infty$. ∎

Theorem 2. (**The inequality of Bernstein, 1972**). *For a nonzero finitely generated module M over the Weyl algebra A_n,*

$$\text{GK}(M) \geq n. ∎$$

Let X be a smooth irreducible algebraic variety of dimension n. Let $\mathcal{D}(X)$ be the ring of differential operators on X.

Example. $X = K^n$, $\mathcal{D}(K^n) = A_n$;

$X = S^n := \{(x_i) \in K^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$, $\mathcal{D}(S^n)$.

The alg. $\mathcal{D}(X)$ is a simple affine Noetherian inf. dim. algebra with $\text{GK}(\mathcal{D}(X)) = 2n$.

Theorem 3. (**McConnell-Robson**). *For a nonzero finitely generated* $\mathcal{D}(X)$*-module* M,

$$\text{GK}(M) \geq n..$$

Question. How to find (estimate) the number

$$i_A := \inf\{\text{GK}(M),\ M \text{ is a nonzero}$$

$$\text{finitely generated } A-\text{module}\}?$$

To find an approximation for i_A was a main **motivation** for introducing the filter dimension.

Let A be a simple affine algebra with the filtration $F = \{A_i\}$, $A = \cup_{i \geq 0} A_i$. Define the **return function** $\nu_F : \mathbf{N} \to \mathbf{N}$ of A:

$\nu_F(i) := \inf\{j \in \mathbf{N} : \ A_j a A_j \ni 1 \text{ for all } 0 \neq a \in A_i\}$,

where $A_j a A_j$ is the subspace of A generated by the products xay, for all $x, y \in A_j$.

Definition. (B., 1996). The **filter dimension** of A:

$$\text{fdim}\, A := \gamma(\nu_F).$$

The filter dimension does not depend on the choice of F.

3. Analog of the Inequality of Bernstein for Simple Affine Algebras

Theorem 4. (B., 1996). *Let A be a simple affine infinite dimensional algebra.*

1. $\text{fdim}\, A \geq 1/2$.

2. *For every nonzero finitely generated $A-$module M:*

$$\text{GK}(M) \geq \frac{\text{GK}(A)}{\text{fdim}(A) + \max\{\text{fdim}(A), 1\}}.$$

Proof. 2. Let $A = K < x_1, \ldots, x_n >= \cup_{i \geq 0} A_i$ and let $F = \{A_i\}$ be the filtration of A.

Let M_0 be a fin. dim. gen. subspace of the A-module M:

$$M = \cup_{i\geq 0} M_i, \ \ M_i = A_i M_0, \ \ i \geq 0.$$

It follows from the definition of the return function $\nu = \nu_F$ of A that, for every $0 \neq a \in A_i$, $1 \in A_{\nu(i)} a A_{\nu(i)}$. Now,

$$M_0 = 1M_0 \subseteq A_{\nu(i)} a A_{\nu(i)} M_0 \subseteq A_{\nu(i)} a M_{\nu(i)},$$

so the linear map

$$A_i \to \mathrm{Hom}_K(M_{\nu(i)}, M_{\nu(i)+i}), a \mapsto (m \mapsto am),$$

is injective, hence

$$\dim A_i \leq \dim \mathrm{Hom}_K(M_{\nu(i)}, M_{\nu(i)+i})$$

$$= \dim M_{\nu(i)} \dim M_{\nu(i)+i}.$$

Using the elementary properties of the degree, we have

$$\mathrm{GK}(A) := \gamma(\dim A_i) \leq \gamma(\dim M_{\nu(i)} \dim M_{\nu(i)+i})$$

$$\leq \gamma(\mathsf{dim}\, M_{\nu(i)}) + \gamma(\mathsf{dim}\, M_{\nu(i)+i})$$

$$\leq \gamma(\mathsf{dim}\, M_i)\gamma(\nu)$$

$$+\gamma(\mathsf{dim}\, M_i) \max\{\gamma(\nu), 1 = \gamma(i \to i)\}$$

$$= \mathsf{GK}(M)(\mathsf{fdim}\, A + \max\{\mathsf{fdim}\, A, 1\}),$$

since

$$\mathsf{GK}(M) = \gamma(\mathsf{dim}\, M_i) \text{ and } \mathsf{fdim}\, A = \gamma(\nu).. \blacksquare$$

Theorem 5. (B., 1998). *Let $\mathcal{D}(X)$ be the ring of differential operators on a smooth irreducible algebraic variety X of dimension n. The filter dimension*

$$\text{fdim}\, \mathcal{D}(X) = 1. \blacksquare$$

- (**McConnell-Robson**). *For a nonzero finitely generated $\mathcal{D}(X)$-module M,*

$$\text{GK}(M) \geq n.$$

Proof.

$$\text{GK}(M) \geq \frac{\text{GK}(\mathcal{D}(X))}{\text{fdim}(\mathcal{D}(X)) + \max\{\text{fdim}(\mathcal{D}(X)), 1\}}$$

$$= \frac{2n}{1 + \max\{1, 1\}} = \frac{2n}{2} = n. \blacksquare$$

4. Inequality between Krull, Gelfand-Kirillov and Filter Dimensions for Simple Affine Algebras. Applications to $\mathcal{D}$-modules

K.dim , the **Krull dimension** (in the sense of **Rentschler-Gabriel**, 1967)

Theorem 6 (**Rentschler-Gabriel**, 1967) *Let A_n be the Weyl algebra. Then* $\text{K.dim}\, A_n = n$.

Theorem 7 (**McConnell-Robson**) *Let X be a smooth irreducible algebraic variety of* dim n. *Then* $\text{K.dim}\, \mathcal{D}(X) = n$.

Definition. An algebra S is called **finitely partitive** if, given any fin. gen. S-module M, there is an integer $n > 0$ s. t. for every chain of submodules

$$M = M_0 \supset M_1 \supset \ldots \supset M_m$$

with $\mathrm{GK}(M_i/M_{i+1}) = \mathrm{GK}(M)$, one has $m \leq n$.

Lemma 8. *$\mathcal{D}(X)$ is a finitely partitive alg., and for any fin. gen. $\mathcal{D}(X)$-module M, $\mathrm{GK}(M)$ is a natural number.*

Theorem 9 (B., 1998) Let A be a finitely partitive simple affine algebra with $\mathrm{GK}(A) < \infty$. Suppose that the Gelfand-Kirillov dimension of every finitely generated A-module is a natural number. Then, for any nonzero finitely generated A-module M, the Krull dimension

$$\mathrm{K.dim}(M) \leq \mathrm{GK}(M) - \frac{\mathrm{GK}(A)}{\mathrm{fdim}(A) + \max\{\mathrm{fdim}(A), 1\}}.$$

In particular,

$$\mathrm{K.dim}(A) \leq \mathrm{GK}(A)\left(1 - \frac{1}{\mathrm{fdim}(A) + \max\{\mathrm{fdim}(A), 1\}}\right).$$

• (**McConnell-Robson**). K.dim $\mathcal{D}(X) = n$.

Proof. GK $\mathcal{D}(X) = 2n$ and fdim $\mathcal{D}(X) = 1$.

By Theorem 9,

K.dim $\mathcal{D}(X) \leq 2n(1 - \frac{1}{1+\max\{1,1\}})$

$= 2n(1 - \frac{1}{2}) = \frac{2n}{2} = n$.

K.dim $\mathcal{D}(X) \geq n$, easy. **.**

****

A generalization to AFFINE ALGEBRAS is given in

V. Bavula and T. Lenagan, "A Bernstein-Gabber-Joseph theorem for affine algebras", Proc. Edinburgh Math. Soc. 42 (1999), no.2, 311–332.

(**Faithful** and **Schur** Dimensions)