# Broué's abelian defect group conjecture holds for the Harada-Norton sporadic simple group $\mathsf{HN}$

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**Abstract**

In representation theory of finite groups, there is a well-known and important conjecture due to M. Broué. He conjectures that, for any prime $p$, if a $p$-block $A$ of a finite group $G$ has an abelian defect group $P$, then $A$ and its Brauer corresponding block $B$ of the normaliser $N_G(P)$ of $P$ in $G$ are derived equivalent (Rickard equivalent). This conjecture is called *Broué's abelian defect group conjecture*.

We prove in this paper that Broué's abelian defect group conjecture is true for a non-principal 3-block $A$ with an elementary abelian defect group $P$ of order 9 of the Harada-Norton simple group $\mathsf{HN}$. It then turns out that Broué's abelian defect group conjecture holds for all primes $p$ and for all $p$-blocks of the Harada-Norton simple group $\mathsf{HN}$.

*Keywords:* Broué's conjecture; abelian defect group; Harada-Norton simple group

## 1. Introduction and notation

In representation theory of finite groups, one of the most important and interesting problems is to give an affirmative answer to a conjecture, which was introduced by M. Broué around 1988 [8], and is nowadays called *Broué's Abelian Defect Group Conjecture*. He actually conjectures the following:

**1.1.Conjecture** (Broué's Abelian Defect Group Conjecture) ([8, 6.2.Question] and [23, Conjecture in p.132])**.** *Let $p$ be a prime, and let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting $p$-modular system for all subgroups of a finite group $G$. Assume that $A$ is a block algebra of $\mathcal{O}G$ with a defect group $P$ and that $B$ is a block algebra of $\mathcal{O}N_G(P)$ such that $B$ is the Brauer correspondent of $A$, where $N_G(P)$ is the normaliser of $P$ in $G$. Then, $A$ and $B$ should be derived equivalent (Rickard equivalent) provided $P$ is abelian.*

In fact, a stronger conclusion than **1.1** is expected. If $G$ and $H$ are finite groups and if $A$ and $B$ are block algebras of $\mathcal{O}G$ and $\mathcal{O}H$ (or $kG$ and $kH$) respectively, we say that $A$ and $B$ are *splendidly Rickard equivalent* in the sense of Linckelmann ([39], [40]), where he calls it a *splendid derived equivalence*, see the end of **1.8**. Note that this is the same as that given

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by Rickard in [57] when $A$ and $B$ are the principal block algebras, which he calls a *splendid equivalence.*

**1.2.Conjecture** (Rickard [57], [58, Conjecture 4, in p.193])**.** *Keep the notation, and suppose that $P$ is abelian as in* **1.1**. *Then, there should be a splendid Rickard equivalence between the block algebras $A$ of $\mathcal{O}G$ and $B$ of $\mathcal{O}N_G(P)$.*

There are several cases where the conjectures of Broué **1.1** and Rickard **1.2** are checked. For example we prove that **1.1** and **1.2** are true for the principal block algebra $A$ of an arbitrary finite group $G$ when the defect group $P$ of $A$ is elementary abelian of order 9 (and hence $p = 3$), see [26, (0.2)Theorem]. Then, it may be natural to ask what about the case of non-principal block algebras with the same defect group $P = C_3 \times C_3$. Namely, this paper should be considered as a continuation of such a project, which has already been accomplished for several cases in our previous papers for the O'Nan simple group and the Higman-Sims simple group in [29, 0.2.Theorem], for the Held simple group and the sporadic simple Suzuki group in [30, Theorem], and for the Janko's simple group $J_4$ [31, Theorem 1.3], see also [47] and [34]. That is to say, our main theorem of this paper is the following:

**1.3.Theorem.** *Let $G$ be the Harada-Norton simple group $\mathsf{HN}$, and let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting 3-modular system for all subgroups of $G$, see the definition* **1.8** *below. Suppose that $A$ is a non-principal block algebra of $\mathcal{O}G$ with a defect group $P$ which is an elementary abelian group $C_3 \times C_3$ of order 9, and that $B$ is a block algebra of $\mathcal{O}N_G(P)$ such that $B$ is the Brauer correspondent of $A$. Then, $A$ and $B$ are splendidly Rickard equivalent, and hence the conjectures* **1.1** *and* **1.2** *of Broué and Rickard hold.*

As a matter of fact, the main result **1.3** above is obtained by proving the following:

**1.4.Theorem.** *Keep the notation and the assumption as in* **1.3**. *Then, the non-principal block algebra $A$ of $\mathcal{O}G$ with a defect group $P = C_3 \times C_3$ and the principal block algebra $A'$ of $\mathcal{O}\mathsf{HS}$ of the Higman-Sims simple group are Puig equivalent, that is $A$ and $A'$ are Morita equivalent which is realized by a $\Delta P$-projective $p$-permutation $\mathcal{O}[G \times \mathsf{HS}]$-module, in other words, $A$ and $A'$ have isomorphic source algebras as interior $P$-algebras.*

Then, it turns out that, as a corollary to the main result (1.3), we eventually can prove that

**1.5.Corollary.** *Broué's abelian defect group conjecture* **1.1** *and even Rickard's splendid equivalence conjecture* **1.2** *are true for all primes $p$ and for all block algebras of $\mathcal{O}G$ when $G = \mathsf{HN}$.*

**1.6.Starting point and strategy.** A story of the birth of this paper is actually very similar to that of the Janko's simple group $J_4$ which is given in [31, **1.6**]. Namely, relatively recently G. Hiss, J. Müller, F. Noeske and J.G. Thackray [17] have determined the 3-decomposition matrix of the group $\mathsf{HN}$ with defect group $C_3 \times C_3$, see **4.1**. Our starting point for this work was actually to realize that the 3-decomposition matrix for the non-principal block of $\mathsf{HN}$ with an elementary abelian defect group of order 9 is exactly the same as that for the principal 3-block of the Higman-Sims simple group $\mathsf{HS}$. Furthermore, the generalised 3-decomposition matrices of these two blocks are the same. Therefore, it is natural to suspect whether these two 3-block algebras would be Morita equivalent not only over an algebraically closed field $k$ of characteristic 3 but also over a complete discrete valuation ring $\mathcal{O}$ whose residue field is $k$, and we might expect even that they are *Puig equivalent* (we shall give a precise definition of Puig equivalence in **1.8** below). Anyhow, since the two conjectures of Broué and Rickard in **1.1** and **1.2** respectively have been solved for the principal 3-block of $\mathsf{HS}$ in a paper of Okuyama [51] it turns out that Broué's abelian defect group conjecture **1.1**

and Rickard's splendid equivalence conjecture **1.2** shall be solved also for the non-principal 3-block of HN with the same defect group $C_3 \times C_3$.

**1.7.Contents.** In §2, we shall give several fundamental lemmas, which are useful and powerful to prove our main results. In §§3 and 4, we shall investigate 3-modular representations for HN and we shall get trivial source ($p$-permutation) modules which are in the non-principal 3-block $A$ of HN with a defect group $P = C_3 \times C_3$. In §5, we shall list data on Green correspondents of simples in the principal 3-block $A'$ of HS, which are known by a result of [64, Theorem], see [51, Example 4.8]. Finally, in §§6-8, we shall give complete proofs of our main results **1.3**, **1.4** and **1.5**.

To achieve our results, next to theoretical reasoning we have to rely on fairly heavy computations. As tools, we use the computer algebra system GAP [12], to calculate with permutation groups as well as with ordinary and Brauer characters. We also make use of the data library [7], in particular allowing for easy access to the data compiled in [10], [19] and [67], and of the interface [66] to the data library [68]. Moreover, we use the computer algebra system MeatAxe [60] to handle matrix representations over finite fields, as well as its extensions to compute submodule lattices [42], radical and socle series [45], homomorphism spaces and endomorphism rings [44], and direct sum decompositions [43]. We give more detailed comments on the relevant computations in the spots where they enter the picture.

**1.8.Notation.** Throughout this paper, we use the following notation and terminology. Let $A$ be a ring. We denote by $1_A$, $Z(A)$ and $A^\times$ for the unit element of $A$, the centre of $A$ and the set of all units in $A$, respectively. We denote by $\mathrm{rad}(A)$ the Jacobson radical of $A$ and by $\mathrm{rad}^i(A)$ the $i$-th power $(\mathrm{rad}(A))^i$ for any positive integer $i$ while we define $\mathrm{rad}^0(A) = A$. We write $\mathrm{Mat}_n(A)$ for the matrix ring of all $n \times n$-matrices whose entries are in $A$. Let $B$ be another ring. We denote by mod-$A$, $A$-mod and $A$-mod-$B$ the categories of finitely generated right $A$-modules, left $A$-modules and $(A, B)$-bimodules, respectively. We write $M_A$, ${}_AM$ and ${}_AM_B$ when $M$ is a right $A$-module, a left $A$-module and an $(A, B)$-bimodule. However, by a module we mean a finitely generated right module unless otherwise stated. Let $M$ and $N$ be $A$-modules. We write $N|M$ if $N$ is (isomorphic to) a direct summand of $M$ as an $A$-module.

From now on, let $k$ be a field and assume that $A$ is a finite dimensional $k$-algebra. Suppose that $M$ is an $A$-module. Then, we denote by $\mathrm{soc}(M)$ the socle of $M$. We define $\mathrm{soc}_0(M) = 0$ and $\mathrm{soc}_1(M) = \mathrm{soc}(M)$. Then, we define $\mathrm{soc}_i(M)$ by $\mathrm{soc}_i(M)/\mathrm{soc}_{i-1}(M) = \mathrm{soc}(M/\mathrm{soc}_{i-1}(M))$ for any integer $i \geqslant 2$. Similarly, we write $\mathrm{rad}^i(M)$ for $M{\cdot}\mathrm{rad}^i(A)$ for any integer $i \geqslant 0$. By using this, we define $L_i(M)$ by $\mathrm{rad}^{i-1}(M)/\mathrm{rad}^i(M)$ for $i = 1, 2, \cdots$. We call $L_i(M)$ the $i$-th Loewy layer of $M$. We denote by $j(M)$ the Loewy length of $M$, namely $j(M)$ is the least positive integer $j$ satisfying $\mathrm{rad}^j(M) = 0$. We write $P(M)$ and $I(M)$ for a projective cover and an injective hull (envelope) of $M$, respectively, and we write $\Omega$ for the Heller operator (functor), namely, $\Omega M$ is the kernel of the projective cover $P(M) \twoheadrightarrow M$. Dually, $\Omega^{-1}M$ is the cokernel of the injective hull $M \rightarrowtail I(M)$. For simple $A$-modules $S_1, \cdots, S_n$ (some of which are possibly isomorphic) we write that $M = a_1 \times S_1 + \cdots + a_n \times S_n$, as composition factors for positive integers $a_1, \cdots, a_n$ when the set of all composition factors are $a_1$ times $S_1$, $\cdots$, $a_n$ times $S_n$. For an $A$-module $M$ and a simple $A$-module $S$, we denote by $c_M(S)$ the multiplicity of all composition factors of $M$ which are isomorphic to $S$. We write $c(S, T)$ for $c_{P(S)}(T)$ for simple $A$-modules $S$ and $T$, namely, this is so-called the Cartan invariant with respect to $S$ and $T$.

To describe the structure of an $A$-module, we either indicate the radical and socle series, in cases where these series coincide and are sufficient for our analysis, or we draw an Alperin diagram [1]. An $A$-module need not have an Alperin diagram, but if it does then it is a compact way to give a more detailed structural description of the module under consideration; note that the Alperin diagram is closely related to the Hasse diagram of the incidence relation amongst the local submodules in the sense of [46], hence for explicit examples is easily determined using the techniques described in [42]. Note, however, that by giving any kind of diagram an $A$-module in general is not uniquely determined up to isomorphism.

Let $N$ be another $A$-module. Then, $\mathrm{Hom}_A(M,N)$ is the set of all right $A$-module-homomorphisms from $M$ to $N$, which canonically is a $k$-vector space, and we denote by $\mathrm{PHom}_A(M,N)$ the set of all (relatively) projective homomorphisms in $\mathrm{Hom}_A(M,N)$, which is a $k$-subspace of $\mathrm{Hom}_A(M,N)$. Hence, we can define the factor space, that is, we write $\underline{\mathrm{Hom}}_A(M,N)$ for the factor space $\mathrm{Hom}_A(M,N)/\mathrm{PHom}_A(M,N)$. By making use of this, as is well-known, we can construct the stable module category $\underline{\mathrm{mod}}$-$A$, which is a quotient category of mod-$A$ such that the set of all morphisms is given by $\underline{\mathrm{Hom}}_A(M,N)$.

In this paper, $G$ is always a finite group and we fix a prime number $p$. Assume that $(\mathcal{K},\mathcal{O},k)$ is a splitting $p$-modular system for all subgroups of $G$, that is to say, $\mathcal{O}$ is a complete discrete valuation ring of rank one such that its quotient field is $\mathcal{K}$ which is of characteristic zero and its residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ is $k$ which is of characteristic $p$, and that $\mathcal{K}$ and $k$ are splitting fields for all subgroups of $G$. We mean by an $\mathcal{O}G$-lattice a finitely generated right $\mathcal{O}G$-module which is a free $\mathcal{O}$-module. We sometimes call it just an $\mathcal{O}G$-module. Let $X$ be a $kG$-module. Then, we write $X^\vee$ for the $k$-dual of $X$, namely, $X^\vee = \mathrm{Hom}_k(X,k)$ which is again a right $kG$-module via $(x)(\varphi g) = (xg^{-1})\varphi$ for $x \in X$, $\varphi \in X^\vee$ and $g \in G$. Similarly, we write $\chi^\vee$ for the dual (complex conjugate) of $\chi$ for an ordinary character $\chi$ of $G$. Let $H$ be a subgroup of $G$, and let $M$ and $N$ be an $\mathcal{O}G$-lattice and an $\mathcal{O}H$-lattice, respectively. Then, let $M{\downarrow}^G_H = M{\downarrow}_H$ be the restriction of $M$ to $H$, and let $N{\uparrow}^G_H = N{\uparrow}^G$ be the induction (induced module) of $N$ to $G$, that is, $N{\uparrow}^G = (N \otimes_{\mathcal{O}H} \mathcal{O}G)_{\mathcal{O}G}$. Similar for $kG$- and $kH$-modules.

We denote by $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ the sets of all irreducible ordinary and Brauer characters of $G$, respectively. Let $A$ be a block algebra ($p$-block) of $\mathcal{O}G$. Then, we write $\mathrm{Irr}(A)$ and $\mathrm{IBr}(A)$ for the sets of all characters in $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ which belong to $A$, respectively. We often mean by $\mathrm{IBr}(A)$ the set of all non-isomorphic simple $kG$-modules belonging to $A$. For ordinary characters $\chi$ and $\psi$ of $G$, we denote by $(\chi,\psi)^G$ the inner product of $\chi$ and $\psi$ in usual sense. Let $X$ and $Y$ be $kG$-modules. Then, we write $[X,Y]^G$ for $\dim_k[\mathrm{Hom}_{kG}(X,Y)]$. We denote by $k_G$ the trivial $kG$-module. Similar for $\mathcal{O}_G$. For $A$-modules $M$ and $N$ we write $[M,N]^A$ for $\dim_k[\mathrm{Hom}_A(M,N)]$.

We say that $M$ is a *trivial source* (*$p$-permutation*) $kG$-module if $M$ is an indecomposable $kG$-module whose source is $k_Q$, where $Q$ is a vertex of $M$. Let $G'$ be another finite group, and let $V$ be an $(\mathcal{O}G,\mathcal{O}G')$-bimodule. Then we can regard $V$ as a right $\mathcal{O}[G\times G']$-module via $v(g,g') = g^{-1}vg'$ for $v \in V$, $g \in G$ and $g' \in G'$. Similar for $(kG,kG')$-bimodules. We denote by $\Delta G$ the diagonal copy of $G$ in $G\times G$, namely, $\Delta G = \{(g,g) \in G\times G \,|\, g \in G\}$. Let $A$ and $A'$ be block algebras of $\mathcal{O}G$ and $\mathcal{O}G'$, respectively. Then, we say that $A$ and $A'$ are *Puig equivalent* if $A$ and $A'$ have a common defect group $P$ (and hence $P \subseteq G\cap G'$) and if there is a Morita equivalence between $A$ and $A'$ which is induced by an $(A,A')$-bimodule $\mathfrak{M}$ such that, as a right $\mathcal{O}[G\times G']$-module, $\mathfrak{M}$ is a $p$-permutation (trivial source) module and $\Delta P$-projective. Similar for blocks of $kG$ and $kG'$. Due to a result of Puig (and independently of Scott), see [55, Remark 7.5], this is equivalent to a condition that $A$ and $A'$ have source algebras which are isomorphic as interior $P$-algebras, see [40, Theorem 4.1]. For an $(\mathcal{O}G,\mathcal{O}G')$-bimodule $V$ and a common subgroup $Q$ of $G$ and $G'$, we set $V^Q = \{v \in V \mid qv = vq, \forall q \in Q\}$. If $Q$ is a $p$-group, the Brauer construction is defined to be a quotient $V(Q) = V^Q/[\sum_{R\subsetneqq Q}\mathrm{Tr}_R^Q(V^R) + \mathrm{rad}\mathcal{O}{\cdot}V^Q]$ where $\mathrm{Tr}_R^Q$ is the usual trace map. The Brauer homomorphism $\mathrm{Br}_Q : (\mathcal{O}G)^Q \to kC_G(Q)$ is obtained from composing the canonical epimorphism $(\mathcal{O}G)^Q \twoheadrightarrow (\mathcal{O}G)(Q)$ and a canonical isomorphism $(\mathcal{O}G)(Q) \xrightarrow{\approx} kC_G(Q)$.

We say that $A$ and $A'$ are *stably equivalent of Morita type* if there exists an $(A,A')$-bimodule $\mathfrak{M}$ such that ${}_A\mathfrak{M}$ and $\mathfrak{M}_{A'}$ are both projective, ${}_A(\mathfrak{M}\otimes_{A'}\mathfrak{M}^\vee)_A \cong {}_AA_A \oplus$ (projective $(A,A)$-bimodule) and ${}_{A'}(\mathfrak{M}^\vee\otimes_A\mathfrak{M})_{A'} \cong {}_{A'}A'_{A'} \oplus$(projective $(A',A')$-bimodule).

We say that $A$ and $A'$ are *splendidly stably equivalent of Morita type* if $A$ and $A'$ have a common defect group $P$ and the stable equivalence of Morita type is induced by an $(A, A')$-bimodule $\mathfrak{M}$ which is a $p$-permutation (trivial source) $\mathcal{O}[G \times G']$-module and is $\Delta P$-projective, see [40, Theorem 3.1]. We say that $A$ and $A'$ are *Rickard equivalent* if $A$ and $A'$ are derived equivalent, namely, $\mathrm{D}^b(\text{mod-}A)$ and $\mathrm{D}^b(\text{mod-}A')$ are equivalent as triangulated categories. We say that $A$ and $A'$ are *splendidly Rickard equivalent* if $A$ and $A'$ are derived equivalent by a complex $M^\bullet \in \mathrm{C}^b(A\text{-mod-}A')$ and its dual $(M^\bullet)^\vee$ such that each term $M^n$ of $M^\bullet$ is a $\Delta(P)$-projective and $p$-permutation module as an $\mathcal{O}[G \times G']$-module, where $\mathrm{C}^b(A\text{-mod-}A')$ is the category of bounded complexes of finitely generated $(A, A')$-bimodules.

For a positive integer $n$, $\mathfrak{A}_n$ and $\mathfrak{S}_n$ denote the alternating and symmetric group on $n$ letters, $M_n$ denotes the Mathieu group, and $C_n$, $D_n$ and $SD_n$ denote the cyclic group, the dihedral group and the semi-dihedral group of order $n$, respectively. For a subgroup $E$ of $\mathrm{Aut}(G)$, $G \rtimes E$ denotes a semi-direct product such that $G$ is normal in $G \rtimes E$ and $E$ acts on $G$ canonically. For $g \in G$ and a subset $S$ of $G$, we denote $g^{-1}Sg$ by $S^g$, and similarly, $x^g = g^{-1}xg$ for $x \in G$. For non-empty subsets $S$ and $T$ of $G$, we write $S =_G T$ if $T = S^g$ for an element $g \in G$.

For other notation and terminology, see the books of Nagao-Tsushima [48] and Thévenaz [62].

## 2. Preliminaries

In this section we list many lemmas, some of which are theorems due to other people. These lemmas are so useful and powerful to prove our main results.

**2.1.Lemma** ([24, (1.1)Lemma]). *Let $A$ be a finite-dimensional algebra over a field and $X$ an $A$-module. Assume that $Y$ is a non-zero uniserial $A$-submodule of $X$ with Loewy layers*

$$\mathrm{rad}^{i-1}(Y)/\mathrm{rad}^i(Y) \cong S_i \qquad \text{for } i = 1, \cdots, n$$

*where $S_i$ is a simple $A$-module. Set $\bar{X} = X/Y$. Then, we get the following:*

(i) *For each $j = 1, \cdots, j(X)$, $\mathrm{rad}^{j-1}(X)/\mathrm{rad}^j(X) \cong \mathrm{rad}^{j-1}(\bar{X})/\mathrm{rad}^j(\bar{X})$ or $\mathrm{rad}^{j-1}(X)/\mathrm{rad}^j(X) \cong \mathrm{rad}^{j-1}(\bar{X})/\mathrm{rad}^j(\bar{X}) \bigoplus S_i$ for some $S_i$.*

(ii) *For each $i = i, \cdots, n$, there is a positive integer $m_i$ such that $m_1 < m_2 < \cdots < m_n$ and that $\mathrm{rad}^{m_i-1}(X)/\mathrm{rad}^{m_i}(X) \cong \Big(\mathrm{rad}^{m_i-1}(\bar{X})/\mathrm{rad}^{m_i}(\bar{X})\Big) \bigoplus S_i$.*

**2.2.Lemma** (Okuyama [50, Lemma 2.2]). *Let $S$ be a simple $kG$-module with vertex $P$, and let $f$ be the Green correspondence with respect to $(G, P, N_G(P))$. If $S$ is a trivial source module, then its Green correspondent $f(S)$ is again simple as $kN_G(P)$-module.*

**2.3.Lemma** (Scott [35, II Theorem 12.4 and I Proposition 14.8] and [5, Corollary 3.11.4]).

(i) *If $M$ is a trivial source $kG$-module, then $M$ uniquely (up to isomorphism) lifts to a trivial source $\mathcal{O}G$-lattice $\widehat{M}$.*

(ii) *If $M$ and $N$ are both trivial source $kG$-modules, then $[M, N]^G = (\chi_{\widehat{M}}, \chi_{\widehat{N}})^G$.*

**2.4.Lemma** (Fong-Reynolds). *Let $H$ be a normal subgroup of $G$, and let $A$ and $B$ be block algebras of $\mathcal{O}G$ and $\mathcal{O}H$, respectively, such that $A$ covers $B$. Let $T = T_G(B)$ be the inertial subgroup (stabiliser) of $B$ in $G$. Then, there is a block algebra $\tilde{A}$ of $\mathcal{O}T$ such that $\tilde{A}$ covers $B$, $1_A 1_{\tilde{A}} = 1_{\tilde{A}} 1_A = 1_{\tilde{A}}$, $A = \tilde{A}^G$ (block induction), and the block algebras $A$ and $\tilde{A}$ are Morita equivalent via a pair $(1_A{\cdot}\mathcal{O}G{\cdot}1_{\tilde{A}},\, 1_{\tilde{A}}{\cdot}\mathcal{O}G{\cdot}1_A)$, that is, the Morita equivalence is a Puig equivalence and induces a bijection*

$$\mathrm{Irr}(\tilde{A}) \to \mathrm{Irr}(A), \quad \tilde{\chi} \mapsto \tilde{\chi}{\uparrow}^G; \qquad \mathrm{Irr}(A) \to \mathrm{Irr}(\tilde{A}), \quad \chi \mapsto \chi{\downarrow}_T{\cdot}1_{\tilde{A}}$$

*between* $\mathrm{Irr}(\tilde{A})$ *and* $\mathrm{Irr}(A)$*, and a bijection*

$$\mathrm{IBr}(\tilde{A}) \to \mathrm{IBr}(A), \quad \tilde{\phi} \mapsto \tilde{\phi}{\uparrow}^{G}; \qquad \mathrm{IBr}(A) \to \mathrm{IBr}(\tilde{A}), \quad \phi \mapsto \phi{\downarrow}_{T}{\cdot}1_{\tilde{A}}$$

*between* $\mathrm{IBr}(\tilde{A})$ *and* $\mathrm{IBr}(A)$*,*

**Proof.** See [30, 1.5.Theorem] and [48, Chapter 5 Theorem 5.10]. ■

**2.5.Lemma.** *Let $A$ be a block algebra of $\mathcal{O}G$ with a defect group $P$, let $N = N_G(P)$, and let $A_N$ be a block algebra of $\mathcal{O}N$ which is the Brauer correspondent of $A$. Moreover, let $(P, e)$ be a maximal $A$-Brauer pair, $H = N_G(P, e)$, the normaliser of $(P, e)$ in $N_G(P)$, and let $B$ be a block algebra of $\mathcal{O}H$ which is the Fong-Reynolds correspondent of $A_N$, see* **2.4***. Then, $A{\downarrow}^{G\times G}_{G\times H}{\cdot}1_B = 1_A{\cdot}\mathcal{O}G{\cdot}1_B$, as a right $\mathcal{O}[G \times H]$-module, has a unique (up to isomorphism) indecomposable direct summand with vertex $\Delta P$.*

**Proof.** See [31, Lemma 2.4] and [48, Chapter 5, Theorem 5.10]. ■

**2.6.Lemma.** *Assume that $G \geqslant H$, and let $A$ and $B$ respectively be block algebras of $\mathcal{O}G$ and $\mathcal{O}H$ with a common defect group $P$, and hence $P \leqslant H$. Suppose, moreover, that a pair $(M, M^\vee)$ induces a splendid stable equivalence of Morita type between $A$ and $B$, where $M$ is an $(A, B)$-bimodule such that $M \,\big|\, 1_A{\cdot}\mathcal{O}G{\cdot}1_B$ as $(A, B)$-bimodules.*

(i) *If $X$ is a non-projective trivial source $kG$-module in $A$, then $(X \otimes_A M)_B = Y \oplus (\mathrm{proj})$ for a non-projective indecomposable $kH$-module $Y$ such that $Y$ has a trivial source.*

(ii) *If $X$ is a non-projective indecomposable $kG$-module in $A$, then $(X \otimes_A M)_B = Y \oplus (\mathrm{proj})$ for a non-projective indecomposable $kH$-module $Y$ such that there is a $p$-subgroup $Q$ of $H$ such that $Q$ is a common vertex of $X$ and $Y$.*

**Proof.** See [31, Lemma 2.7]. ■

**2.7.Lemma.** *Let $k$ be a field, and let $A$ be a finite-dimensional symmetric $k$-algebra. Moreover, suppose that $S$ is a simple $A$-module and $M$ is a projective-free $A$-module. Then, we have $\underline{\mathrm{Hom}}_A(S, M) \cong \mathrm{Hom}_A(S, M)$ and $\underline{\mathrm{Hom}}_A(M, S) \cong \mathrm{Hom}_A(M, S)$ as $k$-spaces.*

**Proof.** Follows by [13, (3.2), (3.2*), (3.3)], see [35, II, Lemma 2.7, Corollary 2.8]. ■

**2.8.Lemma.** *Let $k$ be an algebraically closed field, and let $A$ and $B$ be finite-dimensional symmetric $k$-algebras. Suppose that $M$ is an $(A, B)$-bimodule such that ${}_AM$ and $M_B$ are both projective modules. Then a functor $F : \text{mod-}A \to \text{mod-}B$ defined by $F(X') = X' \otimes_A M$ for $X'_A$, is additive and exact. Assume, furthermore, that $F$ induces a stable equivalence between $A$ and $B$.*

(i) *Let $X$ be a projective-free $A$-module such that $X$ has a simple $A$-submodule $S$. Set $T = F(S)$. Then, we can write $F(X) = Y \oplus R$ for a projective-free $B$-module $Y$ and a projective $B$-module $R$. Now, if $T$ is a simple $B$-module, then we may assume that $Y$ contains $T$ and that $F(X/S) = Y/T \oplus (\mathrm{proj})$.*

(ii) (dual of (i)) *Let $X$ be a projective-free $A$-module such that $X$ has an $A$-submodule $X'$ satisfying that $X/X'$ is simple. Set $T = F(X/X')$. Then, we can write $F(X) = Y \oplus R$ for a projective-free $B$-module $Y$ and a projective $B$-module $R$. Now, if $T$ is a simple $B$-module, then we may assume that $T$ is an epimorphic image of $Y$ and that $\mathrm{Ker}(F(X) \twoheadrightarrow T) = \mathrm{Ker}(Y \twoheadrightarrow T) \oplus (\mathrm{proj})$.*

**Proof.** We get (i) from **2.7** and [30, 1.11.Lemma], just as in the proof of [30, 3.25.Lemma and 3.26.Lemma], see [34, Proposition 2.2]. (ii) is just the dual of (i). ■

**2.9.Lemma** (Linckelmann [37, Theorem 2.1(ii)])**.** *Let $A$ and $B$ be finite-dimensional $k$-algebras for a field $k$ such that $A$ and $B$ are both self-injective and indecomposable as algebras,* *and none of them are simple algebras. Suppose that there is an indecomposable $(A, B)$-bimodule $M$ such that a pair $(M, M^\vee)$ induces a stable equivalence between $A$ and $B$. If $S$ is a simple $A$-module, then $(S \otimes_A M)_B$ is a non-projective indecomposable $B$-module.*

The next lemma is a new result due to Kunugi and the first author. This is actually so useful and convenient when we want to apply so-called "Rouquier's gluing" to our inductive argument in order to get a stable equivalence between two block algebras which we are looking at.

**2.10.Lemma** (Koshitani-Kunugi [28, Theorem 1.2])**.** *Let $A$ be a block algebra of $\mathcal{O}G$ with a cyclic defect group $P \neq 1$. Let $H = N_G(P)$, and let $B$ be a block algebra of $\mathcal{O}H$ such that $B$ is the Brauer correspondent of $A$. Then, we get the following:*

- (i) *The following* (1) *and* (2) *are equivalent:*
  - (1) *The Brauer tree of $A$ is a star with exceptional vertex in the centre, and there exists a non-exceptional irreducible ordinary character $\chi$ of $G$ in $A$ such that $\chi(u) > 0$ for any element $u \in P$.*
  - (2) *The block algebras $A$ and $B$ are Puig equivalent.*
- (ii) *If one of the conditions* (1) *and* (2) *in* (i) *holds (and hence both hold), then all simple $kG$-modules in $A$ are trivial source modules.*
- (iii) *If one of the conditions* (1) *and* (2) *in* (i) *holds (and hence both hold), then there is an indecomposable $(A, B)$-bimodule $\mathfrak{M}$ such that $1_A{\cdot}\mathcal{O}G{\cdot}1_B = \mathfrak{M} \oplus (\mathrm{proj})$ and $\mathfrak{M}$, as an $\mathcal{O}[G \times H]$-module, has $\Delta P$ as its vertex, and $\mathfrak{M}$ realizes a Puig equivalence between $A$ and $B$.*

**2.11.Lemma.** *Let $A$ be a block algebra of $\mathcal{O}G$ with defect group $P$. Set $H = N_G(P)$, and let $B$ be a block algebra of $\mathcal{O}H$ such that $B$ is the Brauer correspondent of $A$. Assume that $Q$ is a subgroup of $P$ with $Q \subseteq Z(G)$. Set $\bar{G} = G/Q$, $\bar{H} = H/Q$ and $\bar{P} = P/Q$. It is well-known that there exist block algebras $\bar{A}$ and $\bar{B}$ of $\mathcal{O}\bar{G}$ and $\mathcal{O}\bar{H}$, respectively, such that $\bar{A}$ and $\bar{B}$ dominate $A$ and $B$, namely $\mathrm{Irr}(\bar{A}) \subseteq \mathrm{Irr}(A)$ and $\mathrm{Irr}(\bar{B}) \subseteq \mathrm{Irr}(B)$, and that both $\bar{A}$ and $\bar{B}$ have $\bar{P}$ as defect groups, see* [48, Chapter 5 Theorems 8.10 and 8.11].

- (i) *It holds that $\bar{H} = N_{\bar{G}}(\bar{P})$ and that $\bar{B}$ is the Brauer correspondent of $\bar{A}$.*

  *In the rest of the lemma, assume in addition that $P$ is elementary abelian of order $p^2$, namely, $P = Q \times R$ with $Q \cong R \cong C_p$.*
- (ii) *It holds that*
  $$\bar{A} \otimes_{\mathcal{O}\bar{H}} \bar{B} = {}_{\bar{A}}(\bar{A}{\cdot}1_{\bar{B}})_{\bar{B}} = {}_{\bar{A}}X_{\bar{B}} \oplus (\mathrm{proj})$$
  *for an indecomposable $(\bar{A}, \bar{B})$-bimodule $X$ with vertex $\Delta\bar{P}$.*
- (iii) *In particular, if $X$ realizes a Morita equivalence between $\bar{A}$ and $\bar{B}$, then there exists an $(A, B)$-bimodule $M$ such that $M$ is an indecomposable direct summand of ${}_A(A{\cdot}1_B)_B$ with vertex $\Delta P$, and hence $M$ induces a Puig equivalence between $A$ and $B$.*

**Proof.** (i) The first part is easy. The second part follows from [49, (3.2)Lemma], see [41, $\ell$.10 on p.1314].

(ii) This follows by [38, Proposition 6.1] since $\bar{P} \cong C_p$.

(iii) This is obtained from **(ii)** and [27, Theorem], see [41, $\ell. - 7 \sim \ell. - 4$ on p.1314] and [40, Theorem 4.1]. ■

**2.12.Lemma.** *Suppose that $p = 3$ and $G = \mathfrak{A}_9$.*

- (i) *There uniquely exists a non-principal block algebra $A$ of $\mathcal{O}G$ with defect group $P \cong C_3$. In addition we can write $\mathrm{Irr}(A) = \{\chi_5, \chi_{17}, \chi_{18}\}$ such that $\chi_5(1) = 27$, $\chi_{17}(1) = 189$, $\chi_{18}(1) = 216$ and $\chi_5(u) = \chi_{17}(u) = 9$ for any element $u \in P - \{1\}$, and that a part of the 3-decomposition matrix is*

| | 27 | 189 |
|---|---|---|
| $\chi_5$ | 1 | 0 |
| $\chi_{17}$ | 0 | 1 |
| $\chi_{18}$ | 1 | 1 |

*where the indices of $\chi_i$ are the same as in* [10, p.37]. *(In the following, we use the notation $A$ and $P$ as in* **(i)***).*

(ii) *Set $H = N_G(P)$. Then $H = (P \times \mathfrak{A}_6).C_2$, where the action on $P \times \mathfrak{A}_6$ by $C_2$ is the diagonal one, extending $\mathfrak{A}_6$ to $\mathfrak{S}_6$.*

(iii) *Let $H$ be as in* **(ii)***, and let $B$ be a block algebra of $\mathcal{O}H$, which is the Brauer correspondent of $A$. Then, $A$ and $B$ are Morita equivalent via an $(A,B)$-bimodule $M$ such that $M$ is (up to isomorphism) the unique indecomposable direct summand of ${}_A(A{\cdot}1_B)_B$ with vertex $\Delta P$, and hence it holds that $M$ induces a Puig equivalence between $A$ and $B$, and that the simples* 27 *and* 189 *in $A$ are both trivial source $kG$-modules.*

**Proof.** (i) This follows from [10, p.37], [67, $A_9$ (mod 3)] and [19].

(ii) Easy by inspection.

(iii) This is obtained from **(i)** and **2.10**. ■

**2.13.Lemma.** *Let $A$ and $B$ be finite dimensional $k$-algebras. Assume that there exists a functor $F : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}B$ realizing a stable equivalence between $A$ and $B$. Assume, in addition, that there is a simple $A$-module $S_0$ such that $S_0$ is sent to a simple $B$-module $T_0$, namely, $F(S_0) = T_0$. Then, for any simple $A$-module $S$ with $S \ncong S_0$, it holds $[F(S), T_0]^B = [T_0, F(S)]^B = 0$.*

**Proof.** We get by **2.7** and the assumptions that

$$\begin{aligned} 0 = \mathrm{Hom}_A(S, S_0) &\cong \underline{\mathrm{Hom}}_A(S, S_0) \\ &\cong \underline{\mathrm{Hom}}_B(F(S), F(S_0)) = \underline{\mathrm{Hom}}_B(F(S), T_0) \\ &\cong \mathrm{Hom}_B(F(S), T_0). \end{aligned}$$

Hence $[F(S), T_0]^B = 0$. The rest is similar. ■

**2.14.Lemma.** *Let $A$ be a finite-dimensional $k$-algebra, and assume that $X$ is an $A$-module satisfying that $(I_1 \oplus I_2) \Big| X$ where $I_1$ and $I_2$ are $A$-submodules of $X$ with $I_1 \cong I_2 \cong I(S)$ for a simple $A$-module $S$. If $Z$ is an $A$-submodule of $X$ such that $\mathrm{soc}(Z)$ is a simple $A$-module, then $X/Z$ has a direct summand isomorphic to $I(S)$.*

**Proof.** Set $S_i = \mathrm{soc}(I_i)$ for $i = 1, 2$. Then, $\mathrm{soc}(I_1 \oplus I_2) = S_1 \oplus S_2 \cong S \oplus S$. Hence $S_1 \oplus S_2 \nsubseteq Z$, so that $S_1 \nsubseteq Z$ or $S_2 \nsubseteq Z$. Thus we may assume $S_1 \nsubseteq Z$. Then, $I_1 \cap Z = 0$ since $I_1$ has a unique minimal $A$-submodule $S_1$. This implies that $X/Z \supseteq (I_1 + Z)/Z \cong I_1/(I_1 \cap Z) \cong I_1$. Hence, $I_1 \hookrightarrow X/Z$. Since $I_1$ is injective, we finally have $I_1 \Big| X/Z$. ■

**2.15.Lemma.** *Let $G$, $H$ and $L$ be finite groups such that all of them contain a common subgroup $P$, namely, $P \subseteq G \cap H \cap L$. Let $M$ be a $k[G \times H]$-module such that $M \Big| k_{\Delta P}{\uparrow}^{G \times H}$, and let $N$ be a $k[H \times L]$-module such that $N \Big| k_{\Delta P}{\uparrow}^{H \times L}$. Then, it follows that $M \otimes_{kH} N \Big| k_{\Delta P}{\uparrow}^{G \times L}$.*

**Proof.** This is a special case of [16, 2.5.Proposition]. ■

**2.16.Lemma.** *Let $A$ be a finite-dimensional $k$-algebra, and assume that $X$ is an indecomposable non-simple $A$-module. Then, it holds $\mathrm{soc}(X) \subseteq \mathrm{rad}(X)$.*

**Proof.** Assume that $\mathrm{soc}(X) \not\subseteq \mathrm{rad}(X)$. Then, $X$ has a simple $A$-submodule $S$ with $S \not\subseteq \mathrm{rad}(X)$. Hence, $X$ has a maximal $A$-submodule $M$ with $S \not\subseteq M$. These imply that $S \cap M = 0$ and $S + M = X$. Namely, $X = S \oplus M$. Since $M$ is indecomposable, $X = S$. This is a contradiction. ■

## 3. 3-Local structure for HN

**3.1.Notation and assumption.** From now on, we assume that $G$ is the Harada-Norton simple group HN, and hence $|G| = 2^{14}{\cdot}3^6{\cdot}5^6{\cdot}7{\cdot}11{\cdot}19 \fallingdotseq 2.7 \times 10^{14}$, see [10, p.164–166] and [14].

**3.2.Lemma.**

(i) *In order to prove Broué's abelian defect group conjecture for $G =$ HN, it suffices to prove it for the case $p = 3$.*

(ii) *There exists a unique 3-block $A$ with non-cyclic abelian defect group $P$, and $P$ is elementary abelian of order 9.*

(iii) *$P$ is the Sylow 3-subgroup of the second largest maximal subgroup 2.HS.2 of $G$, a two-fold cover of the automorphism group of the Higman-Sims simple group HS.*

**Proof.** (i) We may assume $p \in \{2, 3, 5\}$ by **3.1** just as in the proof of [31, Lemma 3.2]. Assume that $p = 2$. Then, $G$ has only two 2-blocks $B_0$ and $B_1$ with positive defect by [67], where $B_0$ is the principal 2-block. Then, the non-principal 2-block $B_1$ has a defect group $D$ with $D \cong SD_{16}$, see [4, Lemma 4.2(c)]. Thus, $B_0$ and $B_1$ both have non-abelian defect groups. Next, suppose $p = 5$. By [67], $G$ has only a unique 5-block $B_0$ which has defect $\geqslant 2$, and hence $B_0$ is the principal 5-block. Then, $B_0$ has non-abelian defect group $5_+^{1+4}.5$ by [10, p.164–166].

(ii) Finally, assume $p = 3$. Sylow 3-subgroups of $G$ are non-abelian by [10, p.164–166]. Thus, $G$ has a unique non-principal 3-block $A$ such that $A$ has a defect group $P$ with $|P| \geqslant 3^2$, and actually $P \cong C_3 \times C_3$, see [4, Lemma 4.2(b)].

(iii) Using the character table of $G$, calculations with GAP [12] show that the conjugacy class 2A of $G$ is a defect class of $A$, where we follow the notation in [10, p.164–166]. Hence $P$ is a Sylow 3-subgroup of the centralizer $C_G(2A) \cong$ 2.HS.2. ■

**3.3.Notation.** From now on, we assume $p = 3$, and we use the notation $A$ and $P$ as in 3.2, namely, $A$ is a block algebra of $kG$ with defect group $P \cong C_3 \times C_3$. Set $H = N_G(P)$, and let $B$ be a block algebra of $kH$ that is the Brauer correspondent of $A$. Let $(P, e)$ be a maximal $A$-Brauer pair in $G$, that it, $e$ is a block idempotent of $kC_G(P)$ such that $\mathrm{Br}_P(1_A){\cdot}e = e$, see [2], [9] and [62, §40]. Set $\widetilde{H} = N_G(P, e)$, namely, $\widetilde{H} = \{g \in N_G(P) \mid e^g = e\}$, where $e^g = g^{-1}eg$. Finally set $E = H/C_G(P)$, and let $Q$ be a subgroup of $P$ of order 3.

**3.4.Lemma.** *It holds the following:*

(i) $H = \widetilde{H} = (P \times \mathfrak{A}_6).SD_{16}$.

(ii) $C_G(P) = C_H(P) = P \times \mathfrak{A}_6$.

(iii) $E = \widetilde{H}/C_G(P) \cong SD_{16}$, *where the action of $E$ on $P$ is given by the embedding of $SD_{16}$ as a Sylow 2-subgroup of* $\mathrm{Aut}(P) \cong \mathrm{GL}_2(3)$.

(iv) *All elements in $P - \{1\}$ are conjugate in $H$, and hence in $G$.*

(v) $P - \{1\} \subseteq$ 3A, *where* 3A *is a conjugacy class of $G$ following the notation in* [10, p.164–166].

(vi) *All subgroups of $P$ of order 3 are conjugate in $H$, and hence in $G$.*

(vii) *Recall the subgroup $Q$ of $P$ in* **3.3**. *Then, we have* $C_G(Q) = Q \times \mathfrak{A}_9$, $N_G(Q) = (Q \times \mathfrak{A}_9).2 \leqslant \mathfrak{A}_{12}$, $C_H(Q) = (P \times \mathfrak{A}_6).2$, *and* $N_H(Q) = (P \times \mathfrak{A}_6).2^2$.

(viii) $C_G(Q)/Q \cong \mathfrak{A}_9$, $C_H(Q)/Q \cong (C_3 \times \mathfrak{A}_6).2$, $N_G(Q)/Q \cong \mathfrak{A}_9.2$, *and* $N_H(Q)/Q \cong (C_3 \times \mathfrak{A}_6).2^2$.

**Proof.** This is found using explicit computation with GAP [12]. The starting point is the smallest faithful permutation representation of $G$ on 1140000 points, available in terms of so-called standard generators [65] in [68]. The associated one-point stabiliser is the largest maximal subgroup $\mathfrak{A}_{12}$ of $G$, which hence can be found explicitly by a randomised Schreier-Sims technique. Having completed that, all the following computations can be done using this permutation representation of $G$.

Actually, one of the standard generators is an element of the $2A$ conjugacy class of $G$, where we use the notation in [10, p.164–166]. Hence the second largest maximal subgroup $2.\mathsf{HS}.2 \cong C_G(2A)$ can be found be a centraliser computation. In turn, by **3.2(iii)** $P$ can be computed explicitly as a Sylow 3-subgroup of $2.\mathsf{HS}.2$.

(i)–(ii) The normaliser $H = N_G(P)$ and the centraliser $C_G(P)$ of $P$ can be computed explicitly, and as these are fairly small groups their structure is easily revealed.

(iii) It follows from [67, $A_6$ (mod 3)] and [19] that $\mathfrak{A}_6$ has exactly two 3-blocks. Let $\beta$ be the non-principal block algebra of $k\mathfrak{A}_6$, and hence $\beta$ is of defect zero. Then, $e = 1_\beta$. Since $\beta$ is a unique block algebra of $k\mathfrak{A}_6$ of defect zero, this shows $H = \widetilde{H}$.

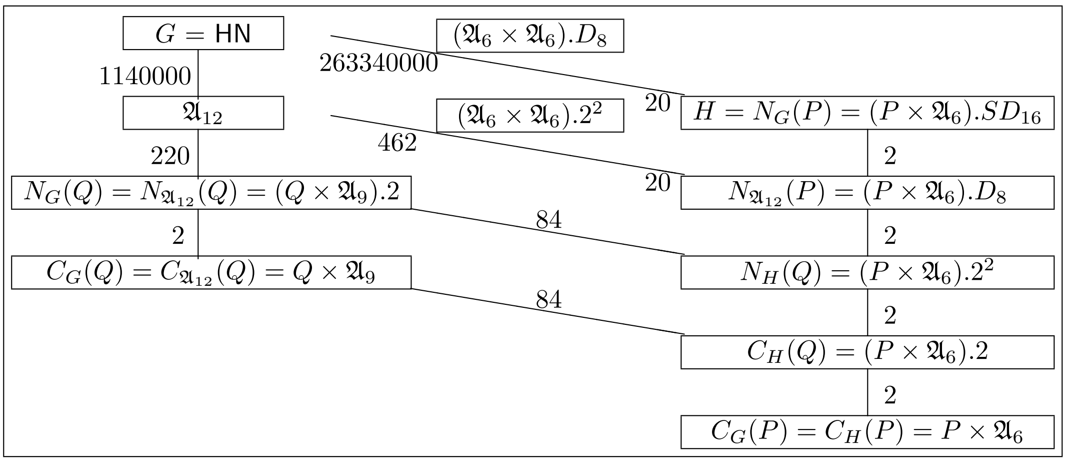
(iv) Easy by **(iii)** and inspection.

(v) We use the notation $3A$ and $3B$ as in [10, p.164–166]. By **(iv)**, $P - \{1\} \subseteq$ 3A or 3B. Assume $P - \{1\} \subseteq$ 3B. Then, $\chi(u) = 0$ for any $\chi \in \mathrm{Irr}(A)$ and any $u \in$ 3A by [48, Chapter 5 Corollary 1.10(i)]. But we know that $\chi_8 \in \mathrm{Irr}(A)$ by [4, Lemma 4.2(b)], see also **4.1**, and that $\chi_8(u) = 27$ for any $u \in$ 3A. This is a contradiction.

(vi) Easy by **(iv)**.

(vii)–(viii) It is easy to see that $N_G(Q) < \mathfrak{A}_{12}$, the largest maximal subgroup of $G$, which is the one-point stabiliser in the given permutation representation of $G$. Hence again the normaliser $N_{\mathfrak{A}_{12}}(Q)$ and the centraliser $C_{\mathfrak{A}_{12}}(P)$ of $P$ can be computed explicitly and their structure determined. ■

**3.5.Lemma.** *We get the following diagram:*




where the numbers between two boxes are indices between the two corresponding groups.

**Proof.** This follows from [10, p.164–166], **3.4** and calculations with GAP [12]. ■

**3.6.Lemma.** *The following holds:*

(i) $B \cong \mathrm{Mat}_9(\mathcal{O}[P \rtimes SD_{16}])$ *as* $\mathcal{O}$*-algebras,*

(ii) *The block algebra $B$ has a source algebra $jBj \cong \mathcal{O}[P \rtimes SD_{16}]$, as interior $P$-algebras, where $j$ is a source idempotent of $B$ with respect to $P$, namely, $j$ is a primitive idempotent of $B^P$ such that $\mathrm{Br}_P(j) \neq 0$ for the Brauer homomorphism $\mathrm{Br}_P$ for $P$, see* [62, §§19 and 27].

(iii) *We can write*

$$\mathrm{Irr}(B) = \{\chi_{9a}, \chi_{9b}, \chi_{9c}, \chi_{9d}, \chi_{18a}, \chi_{18b}, \chi_{18c}, \chi_{72a}, \chi_{72b}\}$$

*and*

$$\mathrm{IBr}(B) = \{9a, 9b, 9c, 9d, 18a, 18b, 18c\},$$

*where the numbers mean the degrees of characters and the dimensions of simples, respectively. Note that $\chi_{18b}$ and $\chi_{18c}$ are dual each other, and so are $18b$ and $18c$. The other characters and simples are self-dual.*

(iv) *The 3-decomposition matrix and the Cartan matrix of $B$ are the following:*

| | $9a$ | $9b$ | $9c$ | $9d$ | $18a$ | $18b$ | $18c$ |
|---|---|---|---|---|---|---|---|
| $\chi_{9a}$ | 1 | . | . | . | . | . | . |
| $\chi_{9b}$ | . | 1 | . | . | . | . | . |
| $\chi_{9c}$ | . | . | 1 | . | . | . | . |
| $\chi_{9d}$ | . | . | . | 1 | . | . | . |
| $\chi_{18a}$ | . | . | . | . | 1 | . | . |
| $\chi_{18b}$ | . | . | . | . | . | 1 | . |
| $\chi_{18c}$ | . | . | . | . | . | . | 1 |
| $\chi_{72a}$ | 1 | 1 | . | . | 1 | 1 | 1 |
| $\chi_{72b}$ | . | . | 1 | 1 | 1 | 1 | 1 |

| | $P(9a)$ | $P(9b)$ | $P(9c)$ | $P(9d)$ | $P(18a)$ | $P(18b)$ | $P(18c)$ |
|---|---|---|---|---|---|---|---|
| $9a$ | 2 | 1 | 0 | 0 | 1 | 1 | 1 |
| $9b$ | 1 | 2 | 0 | 0 | 1 | 1 | 1 |
| $9c$ | 0 | 0 | 2 | 1 | 1 | 1 | 1 |
| $9d$ | 0 | 0 | 1 | 2 | 1 | 1 | 1 |
| $18a$ | 1 | 1 | 1 | 1 | 3 | 2 | 2 |
| $18b$ | 1 | 1 | 1 | 1 | 2 | 3 | 2 |
| $18c$ | 1 | 1 | 1 | 1 | 2 | 2 | 3 |

(v) *There are unique conjugacy classes $4A$ and $4B$ of $H$, consisting of elements of order 4, and having centralisers of order 40 and 48, respectively. A part of the character table of* $\mathrm{Irr}(B)$ *then is the following:*

| conjugacy class | $4A$ | $4B$ | $12A$ |
|---|---|---|---|
| centraliser | 40 | 48 | 24 |
| $\chi_{9a}$ | 1 | $-1$ | $-1$ |
| $\chi_{9b}$ | $-1$ | $-1$ | $-1$ |
| $\chi_{9c}$ | 1 | 1 | 1 |
| $\chi_{9d}$ | $-1$ | 1 | 1 |
| $\chi_{18a}$ | 0 | 0 | 0 |
| $\chi_{18b}$ | 0 | 0 | 0 |
| $\chi_{18c}$ | 0 | 0 | 0 |
| $\chi_{72a}$ | 0 | $-2$ | 1 |
| $\chi_{72b}$ | 0 | 2 | $-1$ |

*Note that this identifies the characters $\chi_{9a}$, $\chi_{9b}$, $\chi_{9c}$, $\chi_{9d}$, $\chi_{72a}$ and $\chi_{72b}$ uniquely.*

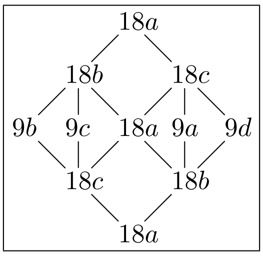
(vi) *The radical and socle series of PIMs in $B$ are the following:*

$$\boxed{\begin{array}{c}9a\\18b\\9b\ 18a\\18c\\9a\end{array}}\quad\boxed{\begin{array}{c}9b\\18c\\9a\ 18a\\18b\\9b\end{array}}\quad\boxed{\begin{array}{c}9c\\18c\\9d\ 18a\\18b\\9c\end{array}}\quad\boxed{\begin{array}{c}9d\\18b\\9c\ 18a\\18c\\9d\end{array}}$$

$$\boxed{\begin{array}{c}18a\\18b\ 18c\\9b\ 9c\ 18a\ 9a\ 9d\\18c\ 18b\\18a\end{array}}\quad\boxed{\begin{array}{c}18b\\9b\ 18a\ 9c\\18c\ 18b\ 18c\\9a\ 18a\ 9d\\18b\end{array}}\quad\boxed{\begin{array}{c}18c\\9a\ 18a\ 9d\\18b\ 18c\ 18b\\9b\ 18a\ 9c\\18c\end{array}}$$

*Note that this identifies the simples* 18$b$ *and* 18$c$ *uniquely.*

(vii) *An Alperin diagram of the PIM $P(18a)$ is given as follows:*

$P(18a) =$ 


**Proof.** This again relies on computations with GAP [12]. Starting with the explicit restriction of the permutation representation of $G$ to $H$ obtained in **3.4**, we find a faithful permutation representation of $H$ on a small number of points. This then is used to compute the conjugacy classes of $H$, and its ordinary character table using the Dixon-Schneider algorithm.

(i) Since the Schur multiplier of $SD_{16}$ is trivial, see e.g. [25, Proof of Lemma 1.3], we get the assertion by **3.4(i)-(iii)**, [33, A.Theorem].

(ii) This follows by a result of Puig [54, Proposition 14.6] and **(i)**, see [3, Theorem 13] and [62, (45.12)Theorem].

(iii)–(v) Easy from the character table of $H$.

(vi) The radical and socle series have been determined in [63].

(vii) To find the structure of $P(18a)$, we have used the MeatAxe [60] to construct $P(18a)$ explicitly as a matrix representation, from the permutation representation of $H$ obtained above, and subsequently we have used the method described in [42] to compute the whole submodule lattice of $P(18a)$, from which the result follows easily. ■

**3.7.Notation.** We use the notation $\chi_{9a}, \chi_{9b}, \chi_{9c}, \chi_{9d}$, $\chi_{18a}$, $\chi_{18b}$, $\chi_{18c}$, $\chi_{72a}$, $\chi_{72b}$, 9$a$, 9$b$, 9$c$, 9$d$, 18$a$, 18$b$, 18$c$, and also the source idempotent $j$ as in **3.6**.

**3.8.Lemma.** *The block algebra $B$ and its source algebra $k[P \rtimes SD_{16}]$ have exactly 18 trivial source modules. In fact, it holds the following:*

(i) *Seven PIMs: $P(9a)$, $P(9b)$, $P(9c)$, $P(9d)$, $P(18a)$, $P(18b)$, $P(18c)$.*

(ii) *Seven trivial source modules with a vertex $P$ :* 9$a$, 9$b$, 9$c$, 9$d$, 18$a$, 18$b$, 18$c$.

(iii) *Four trivial source modules with vertex $Q \cong C_3$:*

$$\chi_{9a}+\chi_{9b}+\chi_{18a}+\chi_{72a} \quad\leftrightarrow\quad V_1 \;=\; \begin{tikzcd}[column sep=small, row sep=small] 9a \arrow[dr, dash] & & 18a \arrow[dl, dash] \arrow[dr, dash] & & 9b \arrow[dl, dash] \\ & 18b \arrow[dl, dash] \arrow[dr, dash] & & 18c \arrow[dl, dash] \arrow[dr, dash] & \\ 9b & & 18a & & 9a \end{tikzcd}$$

$$\chi_{9c}+\chi_{9d}+\chi_{18a}+\chi_{72b} \quad\leftrightarrow\quad V_2 \;=\; \begin{tikzcd}[column sep=small, row sep=small] 9c \arrow[dr, dash] & & 18a \arrow[dl, dash] \arrow[dr, dash] & & 9d \arrow[dl, dash] \\ & 18c \arrow[dl, dash] \arrow[dr, dash] & & 18b \arrow[dl, dash] \arrow[dr, dash] & \\ 9d & & 18a & & 9c \end{tikzcd}$$

$$\chi_{18b}+\chi_{18c}+\chi_{72a} \quad\leftrightarrow\quad V_3 \;=\; \begin{tikzcd}[column sep=small, row sep=small] & 18b \arrow[dr, dash] & & 18c \arrow[dr, dash] & \\ 9b \arrow[ur, dash] \arrow[dr, dash] & & 18a \arrow[ur, dash] \arrow[dr, dash] & & 9a \\ & 18c \arrow[ur, dash] & & 18b \arrow[ur, dash] & \end{tikzcd}$$

$$\chi_{18b}+\chi_{18c}+\chi_{72b} \quad\leftrightarrow\quad V_4 \;=\; \begin{tikzcd}[column sep=small, row sep=small] & 18b \arrow[dr, dash] & & 18c \arrow[dr, dash] & \\ 9c \arrow[ur, dash] \arrow[dr, dash] & & 18a \arrow[ur, dash] \arrow[dr, dash] & & 9d \\ & 18c \arrow[ur, dash] & & 18b \arrow[ur, dash] & \end{tikzcd}$$

*and all characters $\chi_{V_i}$ realized by $V_i$ has values $\chi_{V_i}(u) = 27$ for any $u \in 3A$, where $3A$ is the unique conjugacy class of $H$ of elements of order 3 on which $\chi_{V_i}$ does not vanish, see* [48, Chapter 5, Corollary 1.10(i)].

**Proof.** These follow from **3.4**, a theorem of Green [48, Chapter 4, Problem 10, p.302] As for (iii), starting again with the permutation representation of $H$, using GAP [12] we compute $N_H(Q)$, use the MeatAxe [60] and the methods described in [43] to find the PIMs of $N_H(Q)/Q$ as direct summands of its regular representation, induce them to $H$, and find the submodule structure of the induced modules using the methods described in [42]. ■

**3.9.Notation.** We use the notation $V_1$, $V_2$, $V_3$, $V_4$ as in **3.8**.

**3.10.Lemma.** *There are no $kH$-modules in $B$ whose radical and socle series are the same and which have the following structure:*

$$\boxed{\begin{array}{c} 18a \\ 18b\ 18c \\ 18a \\ 18b \end{array}} \qquad \boxed{\begin{array}{c} 18a \\ 18b\ 18c \\ 18a \\ 18c \end{array}} \qquad \boxed{\begin{array}{c} 9c \\ 18c \\ 9d \\ 18b \end{array}} \qquad \boxed{\begin{array}{c} 9d \\ 18b \\ 9c \\ 18c \end{array}}$$

(i) (ii) (iii) (iv)

**Proof.** (i) Assume that such a $kH$-module, which we call $M$, exists. There is an epimorphism $\pi : P(18a) \twoheadrightarrow M$. Set $K = \mathrm{Ker}(\pi)$. Then, **3.6(vi)** and **1.1** imply that $K$ has radical and socle series

$$\boxed{\begin{array}{c} 9b\ 9c\ 9a\ 9d \\ 18c \\ 18a \end{array}}$$

Since there does not exist a $kH$-module $\boxed{\begin{array}{c} 9a \\ 18c \end{array}}$ by **3.6(vi)**, we have a contradiction.

(ii) Similar to (i).

(iii) Assume that such a $kH$-module, which we call $M$, exists. There is an epimorphism $\pi : P(9c) \twoheadrightarrow M$. Set $K = \mathrm{Ker}(\pi)$. Then, by **3.6(vi)** we get $K = \boxed{\begin{array}{c} 18a \\ 9c \end{array}}$. This contradicts the structure of $P(9c)$ in **3.6(vi)**.

(iv) Similar to (iii). ■

## 4. 3-Modular representations of HN

**4.1.Theorem** (Hiss-Müller-Noeske-Thackray [17])**.** *The* 3*-decomposition matrix and the Cartan matrix of* $A$ *are the following:*

| degree | [10, p.164–166] | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ |
|---|---|---|---|---|---|---|---|---|
| 8910 | $\chi_8$ | 1 | . | . | . | . | . | . |
| 16929 | $\chi_{10}$ | . | 1 | . | . | . | . | . |
| 270864 | $\chi_{19}$ | . | . | 1 | . | . | . | . |
| 1185030 | $\chi_{32}$ | 1 | 1 | . | 1 | . | . | . |
| 1354320 | $\chi_{33}$ | 1 | . | . | . | 1 | 1 | . |
| 1575936 | $\chi_{37}$ | . | . | 1 | . | . | 1 | . |
| 2784375 | $\chi_{43}$ | 1 | . | 1 | 1 | 1 | 1 | . |
| 4561920 | $\chi_{49}$ | . | . | . | 1 | 1 | . | 1 |
| 4809375 | $\chi_{50}$ | . | 1 | 1 | 1 | . | . | 1 |

| | $P(S_1)$ | $P(S_2)$ | $P(S_3)$ | $P(S_4)$ | $P(S_5)$ | $P(S_6)$ | $P(S_7)$ |
|---|---|---|---|---|---|---|---|
| $S_1$ | 4 | 1 | 1 | 2 | 2 | 2 | 0 |
| $S_2$ | 1 | 3 | 1 | 2 | 0 | 0 | 1 |
| $S_3$ | 1 | 1 | 4 | 2 | 1 | 2 | 1 |
| $S_4$ | 2 | 2 | 2 | 4 | 2 | 1 | 2 |
| $S_5$ | 2 | 0 | 1 | 2 | 3 | 2 | 1 |
| $S_6$ | 2 | 0 | 2 | 1 | 2 | 3 | 0 |
| $S_7$ | 0 | 1 | 1 | 2 | 1 | 0 | 2 |

*where* $S_1, \cdots, S_7$ *are non-isomorphic simple* $kG$*-modules in* $A$ *whose degrees respectively are* 8910*,* 16929*,* 270864*,* 1159191*,* 40338*,* 1305072*,* 3362391*.*

**4.2.Notation.** We use the notation $\chi_8, \chi_{10}, \chi_{19}, \chi_{32}, \chi_{33}, \chi_{37}, \chi_{43}, \chi_{49}, \chi_{50}$ and $S_1, \cdots, S_7$ as in **4.1**.

**4.3.Lemma.**

(i) *All simples* $S_1, \cdots, S_7$ *are self-dual.*

(ii) (Knörr) *All simples* $S_1, \cdots, S_7$ *have* $P$ *as vertices.*

**Proof.** (i) Easy from **4.1**.

(ii) This is a result of Knörr [22, 3.7.Corollary]. ∎

**4.4.Lemma.**

(i) *The heart* $\mathcal{H}(P(S_i)) = \mathrm{rad}(P(S_i))/\mathrm{soc}(P(S_i))$ *is indecomposable as a* $kG$*-module for any* $i = 1, \cdots, 7$.

(ii) $\mathrm{Ext}^1_{kG}(S_i, S_j) = 0$ *for any pair* $(i,j) \in \{(1,2),(1,3),(2,1),(2,2),(2,3),(2,7),(3,1),(3,2),(3,5),(3,7),(4,6),(5,3),(5,5),(5,7),(6,4),(6,6),(7,2),(7,3),(7,5),(7,7)\}$.

**Proof.** (i) This follows by the Cartan matrix of $A$ in **4.1** and results of Erdmann and Kawata, see [11, Theorem 1], [20, Theorem 1.5] and [29, 1.9.Lemma].

(ii) If $\mathrm{Ext}^1_{kG}(S_1, S_2) \neq 0$, then $S_2 | \mathcal{H}(P(S_1))$ since $c_{12} = 1$ by **4.1**, which contradicts to **(i)**. Similar for the others. ∎

**4.5.Lemma.**

(i) *The simple* $S_1$ *is a trivial source module with* $S_1 \leftrightarrow \chi_8$.

(ii) *The simple* $S_2$ *is a trivial source module with* $S_2 \leftrightarrow \chi_{10}$.

(iii) *The simple* $S_3$ *is a trivial source module with* $S_3 \leftrightarrow \chi_{19}$.

**Proof.** (i)–(ii) Let $M = 2.\mathsf{HS}.2$, where $\mathsf{HS}$ is the Higman-Sims simple group, be the second largest maximal subgroups of $G$, see by [10, p.164–166]. Now, a calculation with $\mathsf{GAP}$ [12], using the character tables of $M$ and $G$, shows that $1_M{\uparrow}^G{\cdot}1_A = \chi_8 + \chi_{10}$. Set $X = k_M{\uparrow}^G{\cdot}1_A$. We then get $X = S_1 + S_2$ (as composition factors) by **4.1**. Since $X$, $S_1$, $S_2$ are all self-dual by **4.3(i)**, we obtain $X = S_1 \oplus S_2$.

(iii) Let $M$ be the same as above. There uniquely exists a non-trivial linear character $\chi$ of $M$. Then, a calculation with $\mathsf{GAP}$ [12] shows that $\chi{\uparrow}_M^G{\cdot}1_A = \chi_{19}$. Hence, by **4.1**, $S_3$ is a trivial source module. ∎

**4.6.Lemma.** *There is a trivial source* $kG$*-module in* $A$ *whose vertex is* $Q$ *and whose structure is*

$$\boxed{\begin{array}{cccc} & S_4 & & \\ S_1 & S_2 & S_5 & S_7 \\ & S_4 & & \end{array}} \quad \leftrightarrow \quad \chi_{32} + \chi_{49}.$$

**Proof.** It follows from [10, p.164–166] that the fourth largest maximal subgroup of $G$ is of the form $M = 2^{1+8}_+.(\mathfrak{A}_5 \times \mathfrak{A}_5).2$. Let $P_M \in \mathrm{Syl}_3(M)$. Then $P_M \cong C_3 \times C_3$, but a calculation with $\mathsf{GAP}$ [12], using the character tables of $G$ and $M$, shows that $P_M$ contains elements belonging to the $3B$ conjugacy class of $G$, hence $P_M \not=_G P$ by **3.4(v)**. Clearly, there is a non-trivial $kM$-module $T$ with $\dim_k T = 1$. Set $X = T{\uparrow}_M^G{\cdot}1_A$. Then, $X$ is a direct sum of trivial source $kG$-modules, and a calculation with $\mathsf{GAP}$ [12] shows that $X \leftrightarrow \chi_{32} + \chi_{49}$. Since $P$ is a defect group of $A$, any indecomposable $kG$-module $Y$ with $Y|X$ does not have $P$ as its vertex.

Suppose that $X$ is decomposable. Then, **2.3(i)** implies that $X = Y \oplus Z$ such that $Y \leftrightarrow \chi_{32}$ and $Z \leftrightarrow \chi_{49}$. Hence, **4.1** yields that $Y = S_1 + S_2 + S_4$ (as composition factors). We know by **4.3(ii)** and **4.1** that $S_1$, $S_2$, and $Y$ are all self-dual. If $[Y, S_1]^G \neq 0$, then the self-dualities imply $S_1|Y$, and hence $0 \neq [S_1, Y]^G = (\chi_8, \chi_{32})^G$ from **2.3(ii)** and **4.5(i)**, a contradiction. Hence, $[Y, S_1]^G = [S_1, Y]^G = 0$. Similarly, we obtain $[Y, S_2]^G = [S_2, Y]^G = 0$ by **2.3(ii)** and **4.5(ii)**. This is a contradiction since $Y$ has only three composition factors $S_1$, $S_2$ and $S_4$.

Thus, $X$ is indecomposable. By the decomposition matix of $A$ in **4.1**, $X$ is not a PIM. Thus, the order of a vertex of $X$ is 3, and hence $Q$ is a vertex of $X$ by **3.4(vi)**. Clearly, $X$ is a trivial source $kG$-module in $A$. We know by **4.1** that $X = S_1 + S_2 + 2 \times S_4 + S_5 + S_7$ (as composition factors). Note that $X$, $S_1$, $S_2$, $S_4$, $S_5$, $S_7$ are all self-dual from **4.3(i)**. Then, $[S_i, X]^G = [X, S_i]^G = 0$ for any $i = 1, 2, 5, 7$ since $X$ is indecomposable. Thus, $X/\mathrm{rad}(X) \cong \mathrm{Soc}(X) \cong S_4$. Therefore, again by the self-dualities, it holds that $\mathrm{rad}(X)/\mathrm{Soc}(X) \cong S_1 \oplus S_2 \oplus S_5 \oplus S_7$. ■

**4.7.Lemma.** *There is a trivial source $kG$-module in $A$ which has $Q$ as a vertex and has radical and socle series* $\boxed{\begin{array}{c} S_3 \\ S_6 \\ S_3 \end{array}} \leftrightarrow \chi_{19} + \chi_{37}$.

(**Note:** *We can prove that this module has $Q$ as its vertex, but only later on in* **7.2(ii)**).

**Proof.** First, the third largest maximal subgroup of $G$ is of shape $M = U_3(8).3$, see [10, p.164–166]. Then, a calculation with GAP [12], using the character tables of $G$ and $M$, shows that

(1) $$1_M{\uparrow}^G \cdot 1_A = \chi_{19} + \chi_{32} + \chi_{37} + \chi_{49}.$$

Set $X = k_M{\uparrow}^G \cdot 1_A$, hence $X$ is self-dual and is a direct sum of trivial source $kG$-modules. Then, by the decomposition matrix in **4.1**, we know

(2) $$X = S_1 + S_2 + 2 \times S_3 + 2 \times S_4 + S_5 + S_6 + S_7 \qquad \text{(as composition factors).}$$

If $[X, S_1]^G \neq 0$, then **2.3(ii)** and **4.5(i)** imply that $(\chi_{\hat{X}}, \chi_8)^G = [X, S_1]^G \neq 0$ where $\chi_{\hat{X}}$ is a character afforded by $X$ (see **2.3(i)**), which is a contradiction by (1). Hence, it holds $[X, S_1]^G = [S_1, X]^G = 0$ by the self-dualities in **4.3(i)**. Similarly, by **4.5(ii)-(iii)** and **2.3(ii)**, we know also $[X, S_2]^G = [S_2, X]^G = 0$ and $[X, S_3]^G = [S_3, X]^G = 1$. If $[X, S_5]^G \neq 0$, then (2) and the self-dualities imply that $S_5 | X$, and hence $S_5$ is liftable by **2.3(i)**, which contradicts **4.1**. Hence, $[X, S_5]^G = [S_5, X]^G = 0$ by the self-dualities. Similarly, it holds also that $[X, S_i]^G = [S_i, X]^G = 0$ for $i = 6, 7$. If $[X, S_4]^G = 2$, then it follows from (2) and the self-dualities that $(S_4 \oplus S_4) | X$, and hence $S_4$ is liftable by **2.3(i)**, which contradicts **4.1**. This shows $[X, S_4]^G = [S_4, X]^G \neq 2$. Namely,

(3) $$[S_3, X]^G = [X, S_3]^G = 1,$$

(4) $$[S_4, X]^G = [X, S_4]^G \neq 2,$$

(5) $$[S_i, X]^G = [X, S_i]^G = 0 \quad \text{for } i = 1, 2, 5, 6, 7.$$

Now, **4.6** says that there is a trivial source $kG$-module $Y$ that has radical and socle series

(6) $$Y = \boxed{\begin{array}{cccc} & S_4 & & \\ S_1 & S_2 & S_5 & S_7 \\ & S_4 & & \end{array}} \leftrightarrow \chi_{32} + \chi_{49}.$$

in $A$. Then, by (1), (6) and **2.3(ii)**, we have

(7) $$[Y, X]^G = [X, Y]^G = 2.$$

Then, by (4) and (2), we know that

(8) $$[S_4, X]^G = [X, S_4]^G \leqslant 1.$$

Next, we want to claim that there is a homomorphism $\varphi \in \mathrm{Hom}_{kG}(Y, X)$ with $0 \neq \mathrm{Im}(\varphi) \ncong S_4$. Suppose that any non-zero $\varphi \in \mathrm{Hom}_{kG}(Y, X)$ satisfies that $\mathrm{Im}(\varphi) \cong S_4$. By (7), let $\{\varphi_1, \varphi_2\}$ be a $k$-basis of $\mathrm{Hom}_{kG}(Y, X)$. Then, it follows from Schur's lemma that

$\mathrm{Im}(\varphi_1) \neq \mathrm{Im}(\varphi_2)$, and hence that there exists a direct sum $\mathrm{Im}(\varphi_1) \oplus \mathrm{Im}(\varphi_2) \subseteq X$. This means that $[S_4, X]^G \geqslant 2$, contradicting (8).

Therefore, there is a homomorphism $\varphi \in \mathrm{Hom}_{kG}(Y, X)$ with $0 \neq \mathrm{Im}(\varphi) \ncong S_4$. Then, by (6), we know $\mathrm{Ker}(\varphi) = 0$ since $S_i \nmid \mathrm{soc}(X)$ for $i = 1, 2, 5, 7$ by (5). That is, there is a monomorphism $\varphi : Y \rightarrowtail X$ of $kG$-modules.

Then, just by the dual argument, we know also that there is an epimorphism $\psi : X \twoheadrightarrow Y$ of $kG$-modules. It follows then by (2) and (6) that there is a direct sum $\mathrm{Im}(\varphi) \oplus \mathrm{Ker}(\psi) \subseteq X$, and hence $\mathrm{Im}(\varphi) \oplus \mathrm{Ker}(\psi) = X$. Set $Z = \mathrm{Ker}(\psi)$. We can write $X = Y \oplus Z$. Since $Z = 2 \times S_3 + S_6$ (as composition factors), we get by (5) that $Z = \boxed{\begin{array}{c} S_3 \\ S_6 \\ S_3 \end{array}}$. Hence, it is easy to know from (1) and (6) that $Z$ is a trivial source $kG$-module with $Z \leftrightarrow \chi_{19} + \chi_{37}$. ■

**4.8.Lemma.** *There is a trivial source kG-module in A whose structure is*

$$\boxed{\begin{array}{ccc} S_1 & & S_2 \\ & S_4 & \\ S_1 & & S_2 \end{array}} \quad \leftrightarrow \quad \chi_8 + \chi_{10} + \chi_{32}.$$

(**Note:** *We can prove that this module has Q as its vertex, but only later on in* **7.2(i)**).

**Proof.** By [10, p.91], we have $1_{\mathfrak{A}_{11}}\!\uparrow^{\mathfrak{A}_{12}} = 1_{\mathfrak{A}_{12}} + \widetilde{\chi}_{11}$, where $\widetilde{\chi}_{11} \in \mathrm{Irr}(\mathfrak{A}_{12})$ is of degree 11. It follows from the 3-decomposition matrix of $\mathfrak{A}_{12}$ in [67, $A_{12}$ (mod 3)] and [19] that

$$(9) \qquad k_{\mathfrak{A}_{11}}\!\uparrow^{\mathfrak{A}_{12}} = \boxed{\begin{array}{c} k \\ 10 \\ k \end{array}} \quad \leftrightarrow \quad 1_{\mathfrak{A}_{12}} + \widetilde{\chi}_{11},$$

where 10 is a simple $k\mathfrak{A}_{12}$-module of dimension 10. Set $X = (k_{\mathfrak{A}_{11}}\!\uparrow^{\mathfrak{A}_{12}})\!\uparrow^G \cdot 1_A$. Note that $X$ is a direct sum of trivial source $kG$-modules. Then, we know from a calculation with GAP [12], using the character tables of $G$ and $\mathfrak{A}_{12}$, that

$$(10) \qquad X \leftrightarrow \chi_8 + \chi_{10} + \chi_{32}$$

and

$$(11) \qquad X = 2 \times S_1 + 2 \times S_2 + S_4 \qquad \text{(as composition factors)}.$$

By (7), **2.3(ii)** and **4.5(ii)**, we obtain $[X, S_1]^G = (\chi_{\hat{X}}, \chi_8)^G = 1$. Hence, $[X, S_1]^G = [S_1, X]^G = 1$ by the self-dualities. Similarly, we have $[X, S_2]^G = [S_2, X]^G = 1$. Since $S_4$ is not liftable by **4.1**, $S_4$ is not a trivial source module by **2.3(i)**. This implies that $[X, S_4]^G = [S_4, X]^G = 0$ by (8). These yield

$$(12) \qquad X/\mathrm{rad}(X) \cong \mathrm{Soc}(X) \cong S_1 \oplus S_2.$$

Next, we want to claim that $X$ is indecomposable. Suppose that $X$ is decomposable. By (12), we can write $X = X_1 \oplus X_2$ for $A$-submodules $X_1$ and $X_2$ of $X$ with $\mathrm{soc}(X_i) \cong S_i$ for $i = 1, 2$. If $X_1/\mathrm{rad}(X_1) \ncong S_1$, then (12) shows that $X_1/\mathrm{rad}(X_1) \cong S_2$, and hence we get by (12) and (11) that $X = X_1 \oplus X_2 = \boxed{\begin{array}{c} S_2 \\ S_4 \\ S_1 \end{array}} \oplus \boxed{\begin{array}{c} S_1 \\ S_2 \end{array}}$ or $X = X_1 \oplus X_2 = \boxed{\begin{array}{c} S_2 \\ S_1 \end{array}} \oplus \boxed{\begin{array}{c} S_1 \\ S_4 \\ S_2 \end{array}}$, which is a contradiction by the self-dualities of $X$ and each $S_i$ in **4.4(i)**. This means that

$X_i/\mathrm{rad}(X_i) \cong S_i$ for $i = 1, 2$ by (12). If $X_1$ is simple, then we get by (12) that $X_2$ has radical and socle series which is one of the following three cases:

$$\boxed{\begin{matrix} S_2 \\ S_1\ S_4 \\ S_2 \end{matrix}} \quad \boxed{\begin{matrix} S_2 \\ S_1 \\ S_4 \\ S_2 \end{matrix}} \quad \boxed{\begin{matrix} S_2 \\ S_4 \\ S_1 \\ S_2 \end{matrix}}$$

So we have a contradiction by **4.4(ii)**. Thus, $X_1$ is not simple. Similarly, we know that $X_2$ is not simple. Hence, **2.16** yields that $\mathrm{soc}(X_i) \subseteq \mathrm{rad}(X_i)$ for $i = 1, 2$. Thus, $X = X_1 \oplus X_2 = \boxed{\begin{matrix} S_1 \\ S_4 \\ S_1 \end{matrix}} \oplus \boxed{\begin{matrix} S_2 \\ S_2 \end{matrix}}$ or $X = X_1 \oplus X_2 = \boxed{\begin{matrix} S_1 \\ S_1 \end{matrix}} \oplus \boxed{\begin{matrix} S_2 \\ S_4 \\ S_2 \end{matrix}}$. This is a contradiction by (10), **2.3(i)** and **4.1**.

Therefore $X$ is indecomposable. Hence, we get by (11), (12) and **2.16** that $\mathrm{soc}(X) \subseteq \mathrm{rad}(X)$. Thus we get the structure of $X$ as desired. ■

**4.9.Notation.** In the rest of paper let $f$ be the Green correspondence from $G$ to $H$ with respect to $P$, see [48, Chapter 4 §4].

**4.10.Lemma.** *It holds that $f(S_1) = 9a$.*

**Proof.** It follows from **4.5(i)**, **4.3(ii)** and **2.1** that $f(S_1)$ is a simple $kH$-module in $B$, see **3.4(i)**. Using the ordinary characters afforded by the trivial source $kH$-modules in $B$, see **3.8**, we get the following possible decompositions of $S_1{\downarrow_H}{\cdot}1_B$, by a calculation with GAP [12] using the character tables of $G$ and $H$:

$$S_1{\downarrow_H}{\cdot}1_B \;=\; 9a \bigoplus \Big(7 \times P(9a) \oplus 7 \times P(9b) \oplus 5 \times P(18a) \oplus P(18b) \oplus P(18c)\Big)$$

or

$$S_1{\downarrow_H}{\cdot}1_B \;=\; 9b \bigoplus \Big(8 \times P(9a) \oplus 6 \times P(9b) \oplus 5 \times P(18a) \oplus P(18b) \oplus P(18c)\Big).$$

In particular, $f(S_1) = 9a$ or $f(S_1) = 9b$, and we have to decide which case actually occurs.

To this end, let $M = 2.\mathsf{HS}.2$ be the second largest maximal subgroup of $G$, see **4.5**. By [67, $HS$ (mod 3)] and [19], let $A^-$ be the block algebra of $\mathcal{O}M$ containing the unique non-trivial linear character $\chi$ of $M$. Hence letting $A^+$ and $A'$, see **5.1**, be the principal block algebras of $\mathcal{O}M$ and of $\mathcal{O}\mathsf{HS}$, respectively, we have $A^+ \cong A'$ and an isomorphism $-\otimes\chi\colon A^+ \to A^-$. Moreover, $P$ being a Sylow 3-subgroup of $M$, it is the block defect group of $A^-$, and hence let $B^-$ be the Brauer correspondent of $A^-$ in $N_M(P)$.

Using the smallest faithful permutation representation of $M$ on 1408 points, available in [68], the normaliser $N_M(P)$ and the centraliser $C_M(P)$ of the Sylow 3-subgroup $P$ is easily computed explicitly with GAP [12] and their structure determined, we find $N_M(P) = (P{\times}D_8).SD_{16}$ and $C_M(P) = P{\times}D_8$. Now the conjugacy classes of $N_M(P)$ can be computed, its ordinary character table is found using the Dixon-Schneider algorithm, from which its blocks are determined and $B^-$ is identified.

Then a computation with GAP [12], using the character tables of $G$ and $M$, shows that $S_1{\downarrow_M}{\cdot}1_{A^-} = 22^-$, where the latter denotes the unique simple $A^-$-module of that dimension. Moreover, using the character tables of $M$ and $N_M(P)$, GAP [12] shows that $(22^-){\downarrow_{N_M(P)}}{\cdot}1_{B^-} = \lambda$, where $\lambda$ is a certain linear character; actually, $\lambda$ is the Green correspondent of $22^-$ with respect to $(M, P, N_M(P))$, which must be linear in view of **5.7**. Hence $\lambda = (S_1{\downarrow_M}{\cdot}1_{A^-}){\downarrow_{N_M(P)}}{\cdot}1_{B^-}$ is a direct summand of

$$\begin{aligned}(S_1{\downarrow_H}){\downarrow_{N_M(P)}}{\cdot}1_{B^-} &= \Big(f(S_1) \oplus (\mathcal{R}\text{-proj})\Big){\downarrow_{N_M(P)}}{\cdot}1_{B^-} \\ &= f(S_1){\downarrow_{N_M(P)}}{\cdot}1_{B^-} \oplus (Q\text{-proj}) \oplus (\text{proj}),\end{aligned}$$

where $\mathcal{R}$ consists of elementary-abelian 3-subgroups of $H$, of order at most 9, not $H$-conjugate to $P$, and $Q \cong C_3$ is as in **3.3**. Since $\lambda$ has $P$ as a vertex, we conclude that $\lambda$ is a direct summand of $f(S_1){\downarrow_{N_M(P)}}\cdot 1_{B^-}$.

Now a computation with GAP [12], using the character tables of $H$ and $N_M(P)$, shows that $f(S_1){\downarrow_{N_M(P)}}\cdot 1_{B^-} = (9x){\downarrow_{N_M(P)}}\cdot 1_{B^-}$, where $x \in \{a, b\}$, already is linear, where

$$(9a){\downarrow_{N_M(P)}}\cdot 1_{B^-} = \lambda \neq (9b){\downarrow_{N_M(P)}}\cdot 1_{B^-}.$$

This shows that $f(S_1) = 9a$. ∎

**4.11.Lemma.** *It holds that $f(S_2) = 9b$.*

**Proof.** It follows from **4.5(i)**, **4.3(ii)** and **2.1** that $f(S_1)$ is a simple $kH$-module in $B$, see **3.4(i)**. Using the ordinary characters afforded by the trivial source $kH$-modules in $B$, see **3.8**, we get the following possible decompositions of $S_1{\downarrow_H}\cdot 1_B$, by a calculation with GAP [12] using the character tables of $G$ and $H$:

$$S_2{\downarrow_H}\cdot 1_B \;=\; 9b \bigoplus \Big(9 \times P(9a) \oplus 8 \times P(9b) \oplus 7 \times P(18a) \oplus 5 \times P(18b) \oplus 5 \times P(18c)\Big)$$

or

$$S_2{\downarrow_H}\cdot 1_B \;=\; 9a \bigoplus \Big(8 \times P(9a) \oplus 9 \times P(9b) \oplus 7 \times P(18a) \oplus 5 \times P(18b) \oplus 5 \times P(18c)\Big).$$

In particular, $f(S_2) = 9b$ or $f(S_2) = 9a$, hence the assertion follows from **4.10**. ∎

**4.12.Lemma.** *It holds that $f(S_3) = 9c$.*

**Proof.** It follows from **4.5(i)**, **4.3(ii)** and **2.1** that $f(S_1)$ is a simple $kH$-module in $B$, see **3.4(i)**. Using the ordinary characters afforded by the trivial source $kH$-modules in $B$, see **3.8**, we get the following possible decompositions of $S_1{\downarrow_H}\cdot 1_B$, by a calculation with GAP [12] using the character tables of $G$ and $H$:

$$\begin{aligned} S_3{\downarrow_H}\cdot 1_B \;=\; 9c \bigoplus \Big(&54 \times P(9a) \oplus 54 \times P(9b) \oplus 40 \times P(9c) \oplus 41 \times P(9d) \\ &\oplus 94 \times P(18a) \oplus 95 \times P(18b) \oplus 95 \times P(18c)\Big) \bigoplus V_3 \end{aligned}$$

or

$$\begin{aligned} S_3{\downarrow_H}\cdot 1_B \;=\; 9c \bigoplus \Big(&54 \times P(9a) \oplus 54 \times P(9b) \oplus 39 \times P(9c) \oplus 40 \times P(9d) \\ &\oplus 93 \times P(18a) \oplus 96 \times P(18b) \oplus 96 \times P(18c)\Big) \bigoplus V_2 \end{aligned}$$

or

$$\begin{aligned} S_3{\downarrow_H}\cdot 1_B \;=\; 9d \bigoplus \Big(&54 \times P(9a) \oplus 54 \times P(9b) \oplus 41 \times P(9c) \oplus 40 \times P(9d) \\ &\oplus 94 \times P(18a) \oplus 95 \times P(18b) \oplus 95 \times P(18c)\Big) \bigoplus V_3 \end{aligned}$$

or

$$\begin{aligned} S_3{\downarrow_H}\cdot 1_B \;=\; 9d \bigoplus \Big(&54 \times P(9a) \oplus 54 \times P(9b) \oplus 40 \times P(9c) \oplus 39 \times P(9d) \\ &\oplus 93 \times P(18a) \oplus 96 \times P(18b) \oplus 96 \times P(18c)\Big) \bigoplus V_2, \end{aligned}$$

where $V_3$ and $V_2$ are the trivial source $kH$-modules in $B$ with vertex $Q$ given in **3.8**. In particular, $f(S_3) = 9c$ or $f(S_3) = 9d$, and we have to decide which case actually occurs.

Keeping the notation from **4.10**, we by the proof of **4.5(iii)** have $S_3 = \chi{\uparrow^G}\cdot 1_A$, hence $(\chi{\uparrow^G}\cdot 1_A){\downarrow_H}\cdot 1_B = S_3{\downarrow_H}\cdot 1_B = f(S_3) \oplus (Q\text{-proj}) \oplus (\text{proj})$. Hence $f(S_3)$ is a direct summand of

$$(\chi{\uparrow^G}){\downarrow_H}\cdot 1_B = \bigoplus_g \Big((\chi^g){\downarrow_{M^g \cap H}}\Big){\uparrow^H}\cdot 1_B,$$

where $g$ runs through a set of representatives of the $M$-$H$ double cosets in $G$. Since $f(S_3)$ has $P$ as a vertex, and $P$ is normal in $H$, we only have to look at summands coming from $g \in G$ such that $P \leq M^g \cap H$. But for these $g$ we have $P, P^{g^{-1}} \leq M$, which since $P$ is a Sylow 3-subgroup of $M$ implies the existence of $m \in M$ such that $P^m = P^{g^{-1}}$, hence $h := mg \in H$, and thus $g = m^{-1}h \in MH$, that is, we may assume $g = 1$.

Thus we conclude that $f(S_3)$ is a direct summand of $(\chi{\downarrow_{M\cap H}}){\uparrow^H}\cdot 1_B = (\chi{\downarrow_{N_M(P)}}){\uparrow^H}\cdot 1_B$. Now a computation with GAP [12], using the character tables of $N_M(P)$ and $H$, shows that $(\chi{\downarrow_{N_M(P)}}){\uparrow^H}\cdot 1_B = 9c$ is indecomposable, showing that $f(S_3) = 9c$. ■

**4.13.Remark.** We use the notation as in the proof of **4.12**. We just remark that it is possible, using GAP [12] and specially tailored programs to deal efficiently with permutations on millions of points, to construct the transitive permutation representation of $G$ on 3078000 points, that is, the action of $G$ on the cosets of 2.HS in $G$, where 2.HS is the derived subgroup of $M$, and to use the restriction of this representation to $H$ to show that the first of the four possible decompositions of $S_3{\downarrow_H}\cdot 1_B$ listed above actually occurs. But we will not need this fact.

## 5. Green correspondence for HS

**5.1.Notation and assumption.** In the rest of this paper, we use the following notation, too. Let $G'$ be the Higman-Sims simple group HS. Since Sylow 3-subgroups of $G'$ are isomorphic to $C_3 \times C_3$, we by abuse of notation let $P$ denote a Sylow 3-subgroup of HS as well. There is exactly one conjugacy class of $G'$ which contain elements of order 3, that is, $P$ has exactly one $G'$-conjugacy class of subgroups of order 3, see [10, p.81]. Let $H' = N_{G'}(P)$, and hence $H' = (P \rtimes SD_{16}) \times 2$, where the action of $SD_{16}$ on $P$ is given by the embedding of $SD_{16}$ as a Sylow 2-subgroup of $\mathrm{Aut}(P) \cong \mathrm{GL}_2(3)$. Let $A'$ and $B'$, respectively, be the principal block algebras of $\mathcal{O}G'$ and $\mathcal{O}H'$.

**5.2.Lemma.**

(i) *The character table of $P \rtimes SD_{16}$ is given as follows:*

| conjugacy class | $1A$ | $2A$ | $2B$ | $3A$ | $4A$ | $4B$ | $6A$ | $8A$ | $8B$ |
|---|---|---|---|---|---|---|---|---|---|
| centraliser | 144 | 16 | 12 | 18 | 8 | 4 | 6 | 8 | 8 |
| $\chi_{1a}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_{1b}$ | 1 | 1 | 1 | 1 | 1 | $-1$ | 1 | $-1$ | $-1$ |
| $\chi_{1c}$ | 1 | 1 | $-1$ | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| $\chi_{1d}$ | 1 | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 | 1 |
| $\chi_{2a}$ | 2 | 2 | 0 | 2 | $-2$ | 0 | 0 | 0 | 0 |
| $\chi_{2b}$ | 2 | $-2$ | 0 | 2 | 0 | 0 | 0 | $\sqrt{-2}$ | $-\sqrt{-2}$ |
| $\chi_{2c}$ | 2 | $-2$ | 0 | 2 | 0 | 0 | 0 | $-\sqrt{-2}$ | $\sqrt{-2}$ |
| $\chi_{8a}$ | 8 | 0 | 2 | $-1$ | 0 | 0 | $-1$ | 0 | 0 |
| $\chi_{8b}$ | 8 | 0 | $-2$ | $-1$ | 0 | 0 | 1 | 0 | 0 |

*Note that this identifies the characters $\chi_{1a}$, $\chi_{1b}$, $\chi_{1c}$, $\chi_{1d}$, $\chi_{8a}$, and $\chi_{8b}$ uniquely.*

(ii) *$B' \cong \mathcal{O}[P \rtimes SD_{16}]$, as interior $P$-algebras and hence $k$-algebras, and we can write that*

$$\mathrm{Irr}(B') = \{1_{H'} = \chi_{1a}, \chi_{1b}, \chi_{1c}, \chi_{1d}, \chi_{2a}, \chi_{2a}, \chi_{2c} = {\chi_{2b}}^\vee, \chi_{8a}, \chi_{8b}\},$$
$$\mathrm{IBr}(B') = \{1a, 1b, 1c, 1d, 2a, 2b, 2c = 2b^\vee\},$$

*where the numbers mean the degrees (dimensions) of characters (modules). In particular, all simple modules $1a, 1b, 1c, 1d, 2a$ in $B'$ except $2b$ and $2c$ are self-dual.*

(iii) *The 3-decomposition and the Cartan matrices of $B'$, respectively, are the following:*

| | $1a$ | $1b$ | $1c$ | $1d$ | $2a$ | $2b$ | $2c$ |
|---|---|---|---|---|---|---|---|
| $\chi_{1a}$ | 1 | . | . | . | . | . | . |
| $\chi_{1b}$ | . | 1 | . | . | . | . | . |
| $\chi_{1c}$ | . | . | 1 | . | . | . | . |
| $\chi_{1d}$ | . | . | . | 1 | . | . | . |
| $\chi_{2a}$ | . | . | . | . | 1 | . | . |
| $\chi_{2b}$ | . | . | . | . | . | 1 | . |
| $\chi_{2c}$ | . | . | . | . | . | . | 1 |
| $\chi_{8a}$ | 1 | 1 | . | . | 1 | 1 | 1 |
| $\chi_{8b}$ | . | . | 1 | 1 | 1 | 1 | 1 |

| | $P(1a)$ | $P(1b)$ | $P(1c)$ | $P(1d)$ | $P(2a)$ | $P(2b)$ | $P(2c)$ |
|---|---|---|---|---|---|---|---|
| $1a$ | 2 | 1 | 0 | 0 | 1 | 1 | 1 |
| $1b$ | 1 | 2 | 0 | 0 | 1 | 1 | 1 |
| $1c$ | 0 | 0 | 2 | 1 | 1 | 1 | 1 |
| $1d$ | 0 | 0 | 1 | 2 | 1 | 1 | 1 |
| $2a$ | 1 | 1 | 1 | 1 | 3 | 2 | 2 |
| $2b$ | 1 | 1 | 1 | 1 | 2 | 3 | 2 |
| $2c$ | 1 | 1 | 1 | 1 | 2 | 2 | 3 |

**Proof.** (i) This is found using explicit computation with GAP [12]. Using the smallest faithful permutation representation of $G'$ on 100 points, available in [68], $P$ can be computed as a Sylow 3-subgroup of $G'$, and hence the normaliser $H' = N_{G'}(P)$ of $P$ is easily determined explicitly. Now the conjugacy classes of $H'$ can be computed, and its ordinary character table is found using the Dixon-Schneider algorithm. Note that there are unique conjugacy classes $2B$ and $4B$ consisting of elements of order 2 and 4, respectively, and having centralisers of order 12 and 4, respectively.

(ii)–(iii) Easy from the character table. ■

**5.3.Notation.** We use the notation $1_{H'} = \chi_{1a}, \chi_{1b}, \chi_{1c}, \chi_{1d}, \chi_{2a}, \chi_{2a}, \chi_{2c} = {\chi_{2b}}^\vee, \chi_{8a}, \chi_{8b}$ and $1a, 1b, 1c, 1d, 2a, 2b, 2c = 2b^\vee$, as in **5.2**. Namely, we can write

$$\begin{aligned} \mathrm{Irr}(B') = \mathrm{Irr}(H') &= \{1_{H'} = \chi_{1a}, \chi_{1b}, \chi_{1c}, \chi_{1d}, \chi_{2a}, \chi_{2a}, \chi_{2c} = {\chi_{2b}}^\vee, \chi_{8a}, \chi_{8b}\}, \\ \mathrm{IBr}(B') = \mathrm{IBr}(H') &= \{1a, 1b, 1c, 1d, 2a, 2b, 2c = 2b^\vee\} \end{aligned}$$

Let $f'$ and $g'$ be the Green correspondences with respect to $(G', P, H')$.

**5.4.Lemma.**

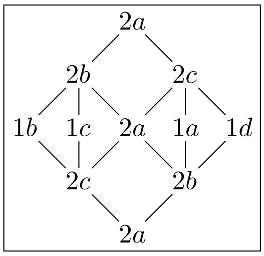
(i) *The radical and socle series of PIMs in $B'$ are the following:*

| | | | |
|---|---|---|---|
| $1a$ | $1b$ | $1c$ | $1d$ |
| $2b$ | $2c$ | $2c$ | $2b$ |
| $1b\ 2a$ | $1a\ 2a$ | $1d\ 2a$ | $1c\ 2a$ |
| $2c$ | $2b$ | $2b$ | $2c$ |
| $1a$ | $1b$ | $1c$ | $1d$ |

| | | |
|---|---|---|
| $2a$ | $2b$ | $2c$ |
| $2b\ 2c$ | $1b\ 2a\ 1c$ | $1a\ 2a\ 1d$ |
| $1b\ 1c\ 2a\ 1a\ 1d$ | $2c\ 2b\ 2c$ | $2b\ 2c\ 2b$ |
| $2c\ 2b$ | $1a\ 2a\ 1d$ | $1b\ 2a\ 1c$ |
| $2a$ | $2b$ | $2c$ |

*Note that this identifies the simples $2b$ and $2c$ uniquely.*

(ii) *An Alperin diagram of the PIM $P(2a)$ is given as follows:*

$P(2a) =$ 


**Proof.** Using the faithful permutation representation of $H'$ obtained in **5.2**, we have used the MeatAxe [60] to construct the PIMs explicitly as matrix representations. Then we have used the method described in [45] to find the radical and socle series, and the method in [42] to compute the whole submodule lattice of $P(2a)$. ■

**5.5.Lemma.**

(i) *We can write that*

$$\mathrm{Irr}(A') = \{\chi'_1 = 1_{G'}, \chi'_{154}, \chi'_{22}, \chi'_{1408}, \chi'_{1925}, \chi'_{770}, \chi'_{3200}, \chi'_{2750}, \chi'_{1750}\}$$
$$\mathrm{IBr}(A') = \{1_{G'}, 154, 22, 1253, 1176, 748, 321\}$$

(ii) *All simples* $1_{G'}, 154, 22, 1253, 1176, 748, 321$ *in* $A'$ *are self-dual, and have* $P$ *as their vertices.*

(iii) *The simples* $1_{G'}, 154, 22$ *are trivial source* $kG'$*-modules.*

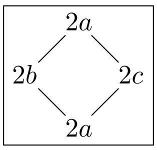
**Proof.** (i) This was first calculated by Humphreys [18, p.329]; see also [67, $HS$ (mod 3)] and [19].

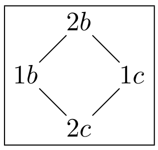
(ii) This is obtained by a result of Knörr [22, 3.7.Corollary].

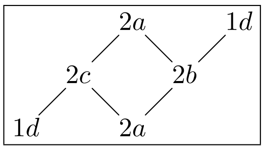
(iii) It follows from [64] or [51, Example 4.8] that the Green correspondents $f'(k_{G'})$, $f'(22)$ and $f'(154)$ are $k_{H'} = 1a$, $1b$ and $1c$, respectively, see **5.7** below. ■

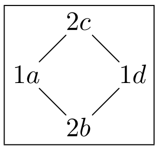
**5.6.Notation.** We write $\chi'_1 = 1_{G'}, \chi'_{154}, \chi'_{22}, \chi'_{1408}, \chi'_{1925}, \chi'_{770}, \chi'_{3200}, \chi'_{2750}, \chi'_{1750}$, as well as $1_{G'}, 154, 22, 1253, 1176, 748, 321$ as in **5.5**.

**5.7.Lemma.** $f'(k_{G'} = 1a) = \boxed{k_{H'} = 1a}$ $\quad f'(154) = \boxed{1b}$ $\quad f'(22) = \boxed{1c}$

$f'(1253) =$ 


$f'(1176) =$ 


$f'(748) =$ 


$f'(321) =$ 


**Proof.** This follows from [64], see [51, Example 4.8, $HS$]. ■

**5.8.Lemma.** *The Cartan matrix of $A'$ is the following:*

| | $P(k_{G'})$ | $P(154)$ | $P(22)$ | $P(1253)$ | $P(1176)$ | $P(748)$ | $P(321)$ |
|---|---|---|---|---|---|---|---|
| $k_{G'}$ | 4 | 1 | 1 | 2 | 2 | 2 | 0 |
| 154 | 1 | 3 | 1 | 2 | 0 | 0 | 1 |
| 22 | 1 | 1 | 4 | 2 | 1 | 2 | 1 |
| 1253 | 2 | 2 | 2 | 4 | 2 | 1 | 2 |
| 1176 | 2 | 0 | 1 | 2 | 3 | 2 | 1 |
| 748 | 2 | 0 | 2 | 1 | 2 | 3 | 0 |
| 321 | 0 | 1 | 1 | 2 | 1 | 0 | 2 |

**Proof.** This was first calculated by Humphreys [18, p.329]; see also [67, *HS* (mod 3)] and [19]. ■

## 6. Stable equivalence between $A$ and $B$ for HN

**6.1.Notation.** First of all, recall the notation $G$, $A$, $P$, $H$, $B$, $e$, $Q$, $E$, $f$ as in **3.3** and **4.9**. Let $i$ and $j$ respectively be source idempotents of $A$ and $B$ with respect to $P$. As remarked in [40, pp.821–822], we can take $i$ and $j$ such that $\mathrm{Br}_P(i){\cdot}e = \mathrm{Br}_P(i) \neq 0$ and that $\mathrm{Br}_P(j){\cdot}e = \mathrm{Br}_P(j) \neq 0$. Set $G_P = C_G(P) = C_H(P) = H_P$, and set $G_Q = C_G(Q)$ and $H_Q = C_H(Q)$. By replacing $e_Q$ and $f_Q$ (if necessary), we may assume that $e_Q$ and $f_Q$ respectively are block idempotents of $kG_Q$ and $kH_Q$ such that $e_Q$ and $f_Q$ are determined by $i$ and $j$, respectively. Namely, $\mathrm{Br}_Q(i){\cdot}e_Q = \mathrm{Br}_Q(i)$ and $\mathrm{Br}_Q(j){\cdot}f_Q = \mathrm{Br}_Q(j)$. Let $A_Q = kC_G(Q){\cdot}e_Q$ and $B_Q = kC_H(Q){\cdot}f_Q$, so that $e_Q = 1_{A_Q}$ and $f_Q = 1_{B_Q}$.

**6.2.Lemma.** *Let $\mathfrak{M}_Q$ be a unique (up to isomorphism) indecomposable direct summand of $A_Q{\downarrow}^{G_Q\times G_Q}_{G_Q\times H_Q}{\cdot}1_{B_Q}$ with vertex $\Delta P$ (note that such an $\mathfrak{M}_Q$ always exists by **2.5**). Then, a pair $(\mathfrak{M}_Q, \mathfrak{M}_Q^\vee)$ induces a Puig equivalence between $A_Q$ and $B_Q$.*

**Proof.** This follows from **3.4(viii)**, **2.12(iii)** and **2.11(iii)**. ■

**6.3.Lemma.**

(i) *The $(A,B)$-bimodule $1_A{\cdot}kG{\cdot}1_B$ has a unique (up to isomorphism) indecomposable direct summand ${}_A\mathfrak{M}_B$ with vertex $\Delta P$. Moreover, a functor $F : \text{mod-}A \to \text{mod-}B$ defined by $X_A \mapsto (X \otimes_A \mathfrak{M})_B$ induces a splendid stable equivalence of Morita type between $A$ and $B$. We use the notation $F$ below as well.*

(ii) *If $X$ is a non-projective trivial source $kG$-module in $A$, then $F(X) = Y \oplus (\mathrm{proj})$ for a non-projective indecomposable $kH$-module $Y$ in $B$ such that $Y$ is also a trivial source module, and $X$ and $Y$ have a common vertex.*

(iii) *If $X$ is a non-projective $kG$-module in $A$, then $F(\Omega X) = \Omega(F(X)) \oplus (\mathrm{proj})$.*

**Proof.** This follows just like in [31, Proof of Lemma 6.3]. Namely, we get the assertion by [2, Proposition 4.21] and [9, Theorem 1.8(i)] for the morphisms in the Brauer categories and also by [32, Theorem], **6.2**, **3.4(vi)** and [40, Theorem 3.1], see [41, Theorem A.1]. ■

**6.4.Notation.** We use the notation $\mathfrak{M}$ and $F$ as in **6.3**.

## 7. Images of simples via the functor $F$

**7.1.Lemma.** *$F(S_1) = 9a$, $F(S_2) = 9b$, $F(S_3) = 9c$.*

**Proof.** These follow from **4.10**, **4.11**, **4.12**, **2.9** and **6.3**. ■

**7.2.Lemma.**

(i) *The trivial source $kG$-module in **4.8** has $Q \cong C_3$ as its vertex.*

(ii) *The trivial source $kG$-module in **4.7** has $Q \cong C_3$ as its vertex.*

**Proof.** (i) Let $X$ be the trivial source $kG$-module in **4.8**. We get by **6.3(ii)** that $F(X) = Y \oplus (\text{proj})$ for a non-projective indecomposable $B$-module $Y$. Then, it follows from **2.7**, **6.3(i)** and **7.1** that

$$\begin{aligned} 0 \neq \mathrm{Hom}_A(X, S_1) &\cong \underline{\mathrm{Hom}}_A(X, S_1) \cong \underline{\mathrm{Hom}}_B(F(X), F(S_1)) \\ &= \underline{\mathrm{Hom}}_B(F(X), 9a) = \underline{\mathrm{Hom}}_B(Y, 9a) \cong \mathrm{Hom}_B(Y, 9a) \end{aligned}$$

as $k$-spaces. Clearly, $Y$ is a trivial source $kH$-module in $B$ by **6.3(ii)**.

Suppose that $X$ has $P$ as a vertex. Then, so does $Y$ by **6.3(ii)**. This yields that $Y \in \{9a, 9b, 9c, 9d, 18a, 18b, 18c\}$ from **3.8**, and hence $Y \in \{9d, 18a, 18b, 18c\}$ by **7.1**. But, the above computation shows that $\mathrm{Hom}_B(Y, 9a) \neq 0$, a contradiction.

Since $X$ is non-projective, we know that $Q$ is a vertex of $X$ from **3.4(vi)**.

(ii) Let $X'$ be the trivial source $kG$-module in **4.7**. We get by **6.3(ii)** that $F(X') = Y' \oplus (\text{proj})$ for a non-projective indecomposable $B$-module $Y'$. Then, it follows from **2.7**, **6.3(i)** and **7.1** that

$$\begin{aligned} 0 \neq \mathrm{Hom}_A(X', S_3) &\cong \underline{\mathrm{Hom}}_A(X', S_3) \cong \underline{\mathrm{Hom}}_B(F(X'), F(S_3)) \\ &= \underline{\mathrm{Hom}}_B(F(X'), 9c) = \underline{\mathrm{Hom}}_B(Y', 9c) \cong \mathrm{Hom}_B(Y', 9c) \end{aligned}$$

as $k$-spaces. Clearly, $Y'$ is a trivial source $kH$-module in $B$ by **6.3(ii)**.

Suppose that $X'$ has $P$ as a vertex. Then, so does $Y'$ by **6.3(ii)**. This yields that $Y' \in \{9a, 9b, 9c, 9d, 18a, 18b, 18c\}$ from **3.8**, and hence $Y' \in \{9d, 18a, 18b, 18c\}$ by **7.1**. But, the above computation shows that $\mathrm{Hom}_B(Y', 9c) \neq 0$, a contradiction.

Since $X'$ is non-projective, we know that $Q$ is a vertex of $X'$ from **3.4(vi)**.

■

**7.3.Lemma.** *Let $X$ be the trivial source $kG$-module with vertex $Q$ showing up in* **4.8** *and* **7.2(i)**. *Then, $F(X) = V_1 \oplus (\text{proj})$, where $V_1$ is the trivial source $kH$-module in $B$ with vertex $Q$ given in* **3.8(iii)**. *Namely,*

$$F\left(\boxed{\begin{array}{ccc} S_1 & & S_2 \\ & S_4 & \\ S_1 & & S_2 \end{array}}\right) = \boxed{\begin{array}{ccccc} 9a & & 18a & & 9b \\ & 18b & & 18c & \\ 9b & & 18a & & 9a \end{array}} \bigoplus (\text{proj}).$$

**Proof.** We know from the proof of **7.2(i)** that $[Y, 9a]^B \neq 0$. Hence we get the assertion by **3.8(iii)**. ■

**7.4.Lemma.** *Let $X'$ be the trivial source $kG$-module with vertex $Q$ showing up in* **4.7** *and* **7.2(ii)**. *Then, $F(X') = V_2 \oplus (\text{proj})$, where $V_2$ is the trivial source $kH$-module in $B$ with vertex $Q$ given in* **3.8(iii)**. *Namely,*

$$F\left(\boxed{\begin{array}{c} S_3 \\ S_6 \\ S_3 \end{array}}\right) = \boxed{\begin{array}{ccccc} 9c & & 18a & & 9d \\ & 18c & & 18b & \\ 9d & & 18a & & 9c \end{array}} \bigoplus (\text{proj}).$$

**Proof.** We know from the proof of **7.2(ii)** that $[Y', 9c]^B \neq 0$. Hence we get the assertion by **3.8(iii)**. ■

**7.5.Lemma.** *It holds that* $F(S_4) = \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}}$

**Proof.** Let $X$ be the trivial source $kG$-module in $A$ with vertex $Q$ given in **4.8** and **7.2(i)**. By **7.3**, we can write $F(X) = V_1 \oplus (\text{proj})$, where $V_1$ is the trivial source $kH$-module in $B$ given in **3.8(iii)**. Then, since $F(S_1) = 9a$ by **7.1**, it follows from **2.8** that

$$F\left(\boxed{\begin{array}{ccc} S_1 & & S_2 \\ & S_4 & \\ & S_2 & \end{array}}\right) = F(X/S_1) = (V_1/9a) \bigoplus (\text{proj}) = \boxed{\begin{array}{ccccc} 9a & & 18a & & 9b \\ & 18b & & 18c & \\ 9b & & 18a & & \end{array}} \bigoplus (\text{proj}).$$

Similarly, we get by **2.8** that

$$F\left(\boxed{\begin{array}{c} S_2 \\ S_4 \\ S_2 \end{array}}\right) = F\left(\mathrm{Ker}\left[\boxed{\begin{array}{ccc} S_1 & & S_2 \\ & S_4 & \\ & S_2 & \end{array}} \twoheadrightarrow S_1\right]\right)$$

$$\cong \mathrm{Ker}\left(\boxed{\begin{array}{ccccc} 9a & & 18a & & 9b \\ & 18b & & 18c & \\ 9b & & 18a & & \end{array}} \twoheadrightarrow 9a\right) \bigoplus (\text{proj}) = \boxed{\begin{array}{ccccc} & & 18a & & 9b \\ & 18b & & 18c & \\ 9b & & 18a & & \end{array}} \bigoplus (\text{proj}).$$

Then, since $F(S_2) = 9b$ by **7.1**, we similarly obtain by **2.8** that

$$F(S_4) = \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \bigoplus (\text{proj}).$$

Therefore, **6.3(i)** and **2.9** imply the assertion. ■

**7.6.Lemma.** *It holds that* $F(S_6) = \boxed{\begin{array}{ccccc} & & 18a & & 9d \\ & 18c & & 18b & \\ 9d & & 18a & & \end{array}}$.

**Proof.** Let $X' = \boxed{\begin{array}{c} S_3 \\ S_6 \\ S_3 \end{array}}$ in **4.7**, that is, $X'$ is a trivial source $kG$-module in $A$ with vertex $Q$. Then, **7.4** yields that

$$F(X') = \boxed{\begin{array}{ccccc} 9c & & 18a & & 9d \\ & 18c & & 18b & \\ 9d & & 18a & & 9c \end{array}} \bigoplus (\text{proj}).$$

Since $F(S_3) = 9c$ by **7.1**, we obtain the assertion from **6.3(i)** and **2.9** just as in the proof of **7.5**. ■

**7.7.Notation.** We use the notation $W = F(S_5) \oplus F(S_7)$ in the rest of this paper.

**7.8.Lemma.** *We get the following.*

(i) *The module $W$ is self-dual.*
(ii) *The module $W$ is a direct sum of exactly two non-projective non-simple indecomposable $B$-modules, and both of them are self-dual.*
(iii) *It holds that $F(S_5)$ and $F(S_7)$ are neither simple $B$-modules, and $2 \leqslant j(W) \leqslant 4$.*
(iv) $[9x, W]^B = [W, 9x]^B = 0$ *for any* $x \in \{a, b, c\}$.

(v) $[18a, W]^B = [W, 18a]^B = 0$.

**Proof.** (i) This follows from **4.3(i)** and **6.3(i)**.

(ii) This follows from **2.9**, **6.3(i)**, **4.3(i)**, **2.3(i)** and **4.1**.

(iii) By **(ii)** and **3.6(vi)**, we get $j(W) \leqslant 4$. Assume that $F(S_5)$ is simple. Then, we know by **3.8(ii)** and **6.3(ii)** that $S_5$ is a trivial source module, and hence $S_5$ lifts to a trivial source $\mathcal{O}G$-module by **2.3(i)**. This contradicts the 3-decomposition matrix in **4.1**. Hence, $F(S_5)$ is not simple. Similarly, we know that $F(S_7)$ is not simple. These imply $j(W) \geqslant 2$.

(iv) This is obtained by **7.1** and **2.13**.

(v) Set $X = F(S_4)$. By **7.5**, there is an epimorphism $X \twoheadrightarrow 18a$. Hence, we get from **2.7** and **7.5** that

$$\begin{aligned}
\mathrm{Hom}_B(18a, W) &\cong \underline{\mathrm{Hom}}_B(18a, W) \ \cong\ \underline{\mathrm{Hom}}_A\Big(F^{-1}(18a), F^{-1}(W)\Big) \\
&= \underline{\mathrm{Hom}}_A\Big(F^{-1}(18a), S_5 \oplus S_7\Big) \ \cong\ \mathrm{Hom}_A\Big(F^{-1}(18a), S_5 \oplus S_7\Big) \\
&\subseteq \mathrm{Hom}_A\left(F^{-1}(X), S_5 \oplus S_7\right) \ =\ \mathrm{Hom}_A(S_4, S_5 \oplus S_7) \ =\ 0. \quad \blacksquare
\end{aligned}$$

**7.9.Notation.** Let $M = \boxed{\begin{array}{ccccccc} & & & S_4 & & & \\ S_1 & & S_2 & & S_5 & & S_7 \\ & & & S_4 & & & \end{array}}$ be the trivial source $kG$-module in $A$ showing up in **4.6**, and set $\mathfrak{X}_B = F(M)$ and we use the notation $\mathfrak{X}$ in the rest of this paper.

**7.10.Lemma.**

(i) *The module* $\mathfrak{X}$ *has a filtration*

$$\mathfrak{X} \ = \ \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \ \Big| \ 9a \oplus 9b \oplus W \ \Big| \ \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}}$$

*namely,* $\mathfrak{X}$ *has submodules* $\mathfrak{X} \supsetneqq Y \supsetneqq Z$ *such that* $\mathfrak{X}/Y \cong Z \cong \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}}$ *and* $Y/Z \cong 9a \oplus 9b \oplus W$.

(ii) *It holds* $\mathfrak{X} = V \oplus P(18a)$ *where* $V \in \{V_3, V_4\}$.

**Proof.** (i) This follows from **4.6**, **7.1** and **7.5**.

(ii) We know by **6.3(ii)** that $\mathfrak{X} = V \oplus L$ for an indecomposable $kH$-module $V$ in $B$ with vertex $Q$ and a projective $kH$-module $L$ in $B$. Note that $V_i \not| \mathfrak{X}$ for $i = 1, 2$ by **7.3** and **7.4**. Thus, $V \in \{V_3, V_4\}$ by **3.8(iii)**. Moreover, since $[V_3, 18a]^B = [V_4, 18a]^B = 0$ by **3.8(iii)**, we know that $[V, 18a]^B = [V, 18a]^B = 0$ again by **3.8(iii)**. Thus, we have $P(18a)|L$ by **(i)**, and hence $P(18a)|\mathfrak{X}$.

Next, assume that $P(T)|L$ for a simple $kH$-module $T$ in $B$ with $T \ncong 18a$. Since $Z$ has a unique minimal submodule, and which is isomorphic to $18a$, we have that $P(T) \cap Z = 0$ in $\mathfrak{X}$, and hence that there is a direct sum $P(T) \oplus Z$ in $\mathfrak{X}$. Set $\bar{\mathfrak{X}} = \mathfrak{X}/Z$. Clearly, $\bar{\mathfrak{X}} \supseteq$

$(P(T) \oplus Z)/Z \cong P(T)$. Since $P(T)$ is injective, it holds $P(T)|\bar{\mathfrak{X}}$. Set $U = (\bar{\mathfrak{X}})^\vee$. Then, by the dualities, we know $P(T^\vee)|U$. Now, by the filtration of $\mathfrak{X}$, $U$ has a filtration

$$U = 9a \oplus 9b \oplus W \;\Bigg|\; \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}}$$

Namely, $U$ has a submodule $Z'$ such that

$$Z' \cong \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \quad \text{and} \quad U/Z' \cong 9a \oplus 9b \oplus W.$$

We have $T^\vee \not\cong 18a$ by **3.6(iii)**. Hence, we get $P(T^\vee) \cap Z' = 0$ in $U$, and hence there is a direct sum $P(T^\vee) \oplus Z' \subseteq U$. Then, we have

$$P(T^\vee) \cong (P(T^\vee) \oplus Z')/Z' \subseteq U/Z' \cong 9a \oplus 9b \oplus W.$$

Since $P(T^\vee)$ is injective, it holds that $P(T^\vee)|(9a \oplus 9b \oplus W)$, so that $P(T^\vee)|W$ by **3.6(vi)**. This is a contradiction by **7.8(ii)**.

Now, assume that $(P(18a) \oplus P(18a))|\mathfrak{X}$. Then, since $\mathrm{soc}(Z) \cong 18a$, it follows from **2.14** that

$$P(18a) \;\Big|\; \mathfrak{X}/Z \;=\; \left( \boxed{\begin{array}{ccc} & 18a & \\ 18b & & 18c \\ & 18a & \end{array}} \;\Bigg|\; 9a \oplus 9b \oplus W \right).$$

Then, by taking its dual, we get also that

$$P(18a) \;\Big|\; (\mathfrak{X}/Z)^\vee \;=\; \left( 9a \oplus 9b \oplus W \;\Big|\; Z \right)$$

where the right-hand-side is a filtration, by using **7.8(i)** and **3.6(iii)**. Set $N = (\mathfrak{X}/Z)^\vee$. Then, we may consider that $N$ has a $B$-submodule $Z$ such that $N/Z \cong 9a \oplus 9b \oplus W$ and $N = P(18a) \oplus N'$ for a $B$-submodule $N'$ of $N$. Since $j(Z) = 3$, it holds $Z \subseteq \mathrm{soc}_3(N) = \mathrm{soc}_3(P(18a)) \oplus \mathrm{soc}_3(N')$. This implies that there exists a $B$-epimorphism $\pi : N/Z \twoheadrightarrow N/\mathrm{soc}_3(N)$. Clearly,

$$\begin{aligned} N/\mathrm{soc}_3(N) &= [P(18a) \oplus N']/[\mathrm{soc}_3(P(18a)) \oplus \mathrm{soc}_3(N')] \\ &\cong [P(18a)/\mathrm{soc}_3(P(18a)] \oplus [N'/\mathrm{soc}_3(N')]. \end{aligned}$$

Since $P(18a)/\mathrm{soc}_3(P(18a)) = \boxed{\begin{array}{c} 18a \\ 18b\ 18c \end{array}}$ by **3.6(vi)**, we get that $18a \,|\, [(N/Z)/\mathrm{rad}(N/Z)] \cong 9a \oplus 9b \oplus [W/\mathrm{rad}(W)]$. This shows that $[W, 18a]^B \neq 0$, which is a contradiction by **7.8(v)**.

Thus, we know $[P(18a)|L]^B = 1$. Therefore, we get $L \cong P(18a)$. We are done. ■

**7.11.Lemma.** $W/\mathrm{rad}(W) \cong \mathrm{soc}(W) \cong 18b \oplus 18c$.

**Proof.** By **7.8(i)** and **3.6(iii)**, it suffices to show only $W/\mathrm{rad}(W) \cong 18b \oplus 18c$. By **7.10(ii)**, we have

$$(13)\quad \mathfrak{X} \;=\; V \oplus P(18a) \;=\; \boxed{\begin{array}{ccccccc} & 18b & & & 18c & & \\ & \diagup \;\; \diagdown & & & \diagup \;\; \diagdown & & \\ 9x & & & 18a & & & 9y \\ & \diagdown \;\; \diagup & & & \diagdown \;\; \diagup & & \\ & 18c & & & 18b & & \end{array}} \oplus P(18a), \quad \text{where } (x,y) \in \{(b,a),(c,d)\}.$$

By **7.10(i)**, $\mathfrak{X}$ has a filtration

$$(14)\qquad \mathfrak{X} \;=\; \boxed{\begin{array}{ccc} & 18a & \\ \diagup & & \diagdown \\ 18b & & 18c \\ \diagdown & & \diagup \\ & 18a & \end{array}} \;\Bigg|\; 9a \oplus 9b \oplus W \;\Bigg|\; \boxed{\begin{array}{ccc} & 18a & \\ \diagup & & \diagdown \\ 18b & & 18c \\ \diagdown & & \diagup \\ & 18a & \end{array}}$$

Set $L_i(W) = \mathrm{rad}^i(W)/\mathrm{rad}^{i+1}(W)$ for each $i = 0, 1, \ldots.$ Then (13) and (14) show that $(18b \oplus 18c) | L_1(W)$. Recall $(18b)^\vee \cong 18c$ by **3.6(iii)**.

Suppose that $(18b \oplus 18b) \,|\, L_1(W)$. Then, by (14), $\mathfrak{X}$ has a factor module $\bar{\mathfrak{X}}$ which has a filtration

$$\boxed{\begin{array}{ccc} & 18a & \\ \diagup & & \diagdown \\ 18b & & 18c \\ \diagdown & & \diagup \\ & 18a & \end{array}} \;\Bigg|\; 18b \oplus 18b$$

Since $[\mathfrak{X}, 18b]^B = 1$ by (13), and since there do not exist modules of forms $\boxed{\begin{array}{c} 18b \\ 18b \end{array}}$ nor $\boxed{\begin{array}{c} 18c \\ 18b \end{array}}$ by **3.6(vi)**, there must be a $kH$-module having radical and socle series

$$\boxed{\begin{array}{c} 18a \\ 18b\ 18c \\ 18a \\ 18b \end{array}}$$

But this is a contradiction by **3.10(i)**.

Similarly, we get a contradiction by using **3.10(ii)** if $(18c \oplus 18c) | L_1(W)$.

Thus it holds that $[W, 18b]^B = [W, 18c]^B = 1$ and $[W, T]^B = 0$ for any $T \in \{9a, 9b, 9c, 18a\}$ by **3.6(iii)** and **7.8(iv)-(v)**. However, we have to investigate for $9d$.

Assume, first, that the case $(x, y) = (b, a)$ happens in (13). Then, (14) and (13) imply that $W = 9a + 9b + 9c + 9d + 2 \times 18b + 2 \times 18c$, as composition factors.

Suppose that $9d \,|\, L_1(W)$. Then, since $c_W(9d) = 1$ and since $W$ and $9d$ are both self-dual by **3.6(iii)** and **7.8(i)**, we get that $9d \,|\, W = F(S_5) \oplus F(S_7)$. Recall that $F(S_5)$ and $F(S_7)$ are both non-projective indecomposable $kH$-modules by **2.9** and **6.3(i)-(ii)**. Since $9d$ is a trivial source $kH$-module by **3.8(ii)**, we know by **6.3(ii)** that $S_5$ or $S_7$ is a trivial source module, and hence that $S_5$ or $S_7$ lifts from $k$ to $\mathcal{O}$ by **2.3(i)**. This is a contradiction by the 3-decomposition matrix in **4.1**.

Hence, $9d \nmid L_1(W)$. This yields $L_1(W) \cong 18b \oplus 18c$.

Next, assume that the case $(x, y) = (c, d)$ in (13) happens. Then, (13) and (14) imply that

$$(15)\qquad W = 2 \times 9c + 2 \times 9d + 2 \times 18b + 2 \times 18c, \qquad \text{as composition factors.}$$

Suppose that $(9d \oplus 9d) \,|\, L_1(W)$. Then, the self-dualities of $9d$ and $W$ in **3.6(iii)** and **7.8(i)** imply that $(9d \oplus 9d) \,|\, W = F(S_5) \oplus F(S_7)$. Hence, $W \cong 9d \oplus 9d$ by **7.8(ii)**, contradicting **7.7** and **7.8**.

Thus,

$$[W, 9d]^B \leqslant 1. \tag{16}$$

Assume, next, that $[W, 9d]^B = 1$. Hence, by the dualities in **3.6(iii)**, we have

$$L_1(W) \cong \mathrm{soc}(W) \cong 18b \oplus 18c \oplus 9d. \tag{17}$$

We get by **7.8** that $W = W_1 \oplus W_2$ where $W_i$ is a non-simple non-projective indecomposable self-dual $B$-module for $i = 1, 2$. Thus, by (17) and by interchanging $W_1$ and $W_2$, we may assume that $L_1(W_1) \cong 18b, 18c$ or $9d$.

**Case 1:** $L_1(W_1) \cong 18b$. Then, $\mathrm{soc}(W_1) \cong 18c$ since $(18b)^\vee \cong 18c$ by **3.6(iii)** and since $W_1$ is self-dual. Hence, the structure of $P(18b)$ in **3.6(vi)** yields that $W_1 = \boxed{\begin{array}{c}18b\\9c\\18c\end{array}}$. Hence, (15) and (17) imply that $L_1(W_2) \cong 18c \oplus 9d$ and $L_2(W_2) \cong 9c$. But this is a contradiction since $\mathrm{Ext}^1_B(18c, 9c) = 0 = \mathrm{Ext}^1_B(9d, 9c)$ by **3.6(vi)**.

**Case 2:** $L_1(W_1) \cong 18c$. As in **Case 1**, we know that $W_1 = \boxed{\begin{array}{c}18c\\9c\\18b\end{array}}$. Then we get a contradiction by **3.6(vi)** as in **Case 1**.

**Case 3:** $L_1(W_1) \cong 9d$. By the self-dualities of $W_1$ in **7.8(ii)** and simple $B$-modules in **3.6(iii)**, we get that $\mathrm{soc}(W_1) \cong 9d$. It follows by **2.16** that $\mathrm{soc}(W_1) \subseteq \mathrm{rad}(W_1)$. Hence $c_{W_1}(9d) = 2$ by (15). Thus, the structure of $P(9d)$ in **3.6(vi)** yields that $W_1 \cong P(9d)$, a contradiction.

Therefore $[W, 9d]^B \neq 1$, and hence $[W, 9d]^B = 0$ by (16). So that we have $L_1(W) \cong 18b \oplus 18c$. ■

**7.12.Lemma.** $\mathfrak{X} = V_3 \oplus P(18a)$.

**Proof.** Suppose that $\mathfrak{X} = V_4 \oplus P(18a)$. Then, we get by **7.10(i)-(ii)** and **3.6(iv)** that $W = 2 \times 9c + 2 \times 9d + 2 \times 18b + 2 \times 18c$, as composition factors. We use the same notation $L_i(W)$ as in the proof or **7.11**. By **7.11**, $L_1(W) \cong 18b \oplus 18c$. Since $c_W(9c) = 2$, it follows from **3.6(vi)** and **7.8(iii)** that $j(W) = 4$ and $9c \,|\, L_4(W)$. This means $9c \,|\, \mathrm{soc}(W)$, contradicting **7.11**. Therefore, we get the assertion by **7.10(ii)**. ■

**7.13.Lemma.** $W = \boxed{\begin{array}{ccc} & 18b & \\ \diagup & & \diagdown \\ 9b & & 9c \\ \diagdown & & \diagup \\ & 18c & \end{array}} \oplus \boxed{\begin{array}{ccc} & 18c & \\ \diagup & & \diagdown \\ 9a & & 9d \\ \diagdown & & \diagup \\ & 18b & \end{array}} = F(S_5) \oplus F(S_7).$

*Namely, either one of the following two cases occurs:*

Case (a) : $F(S_5) = \boxed{\begin{array}{ccc} & 18b & \\ \diagup & & \diagdown \\ 9b & & 9c \\ \diagdown & & \diagup \\ & 18c & \end{array}}$ and $F(S_7) = \boxed{\begin{array}{ccc} & 18c & \\ \diagup & & \diagdown \\ 9a & & 9d \\ \diagdown & & \diagup \\ & 18b & \end{array}}$

Case (b) : $F(S_5) = \boxed{\begin{array}{ccc} & 18c & \\ \diagup & & \diagdown \\ 9a & & 9d \\ \diagdown & & \diagup \\ & 18b & \end{array}}$ and $F(S_7) = \boxed{\begin{array}{ccc} & 18b & \\ \diagup & & \diagdown \\ 9b & & 9c \\ \diagdown & & \diagup \\ & 18c & \end{array}}$

**Proof.** Here as well we use the notation $L_i(W)$ for $i = 1, 2, \ldots$ just as in the proof of **7.11**. It follows from **7.10(i)** that $\mathfrak{X}$ has a filtration

$$(18) \qquad \mathfrak{X} \; = \; \boxed{\begin{array}{ccccc} & & 18a & & \\ & / & & \backslash & \\ 18b & & & & 18c \\ & \backslash & & / & \\ & & 18a & & \end{array}} \; \Bigg| \; 9a \oplus 9b \oplus W \; \Bigg| \; \boxed{\begin{array}{ccccc} & & 18a & & \\ & / & & \backslash & \\ 18b & & & & 18c \\ & \backslash & & / & \\ & & 18a & & \end{array}}$$

namely, $\mathfrak{X}$ has submodules $Y$ and $Z$ such that $\mathfrak{X} \supsetneqq Y \supsetneqq Z$, $\mathfrak{X}/Y \cong Z \cong \boxed{\begin{array}{ccccc} & & 18a & & \\ & / & & \backslash & \\ 18b & & & & 18c \\ & \backslash & & / & \\ & & 18a & & \end{array}}$ and $Y/Z \cong 9a \oplus 9b \oplus W$. On the other hand, **7.12** says that

$$(19) \qquad \mathfrak{X} = \boxed{\begin{array}{ccccccccc} & & 18b & & & & 18c & & \\ & / & & \backslash & & / & & \backslash & \\ 9b & & & & 18a & & & & 9a \\ & \backslash & & / & & \backslash & & / & \\ & & 18c & & & & 18b & & \end{array}} \bigoplus P(18a).$$

Then, we know by (18), (19) and **3.6(iv)** that

$$(20) \qquad W = 9a + 9b + 9c + 9d + 2 \times 18b + 2 \times 18c, \quad \text{as composition factors.}$$

By **7.11** and (20), we know $j(W) \geqslant 3$.

Assume that $j(W) \geqslant 4$. Then, $j(W) = 4$ by **7.8(iii)**. Since $L_1(W) \cong 18b \oplus 18c$ by **7.11**, we get by **3.6(vi)** that

$$L_4(W) \; \Big| \; L_4(P(18b)) \bigoplus L_4(P(18c)) \; = \; (9a \oplus 18a \oplus 9d) \bigoplus (9b \oplus 18a \oplus 9c)$$

and

$$L_4(W) \; \Big| \; \mathrm{soc}(W) \; = \; 18b \oplus 18c.$$

This is a contradiction.

Hence $j(W) = 3$. Thus, again by **7.11**, (20) and **3.6(vi)**, we know that $W$ has radical and socle series

$$(21) \qquad \boxed{\begin{array}{c} 18b \; 18c \\ 9b \; 9c \; 9a \; 9d \\ 18c \; 18b \end{array}}$$

Now, as in the proof of **7.11**, we get by **7.8** that $W = W_1 \oplus W_2$ where $W_i$ is a non-simple non-projective indecomposable self-dual $B$-module for $i = 1, 2$. Then, by (21), we may assume that $L_1(W_1) \cong 18b$, $\mathrm{soc}(W_1) \cong 18c$, $L_1(W_2) \cong 18c$ and $\mathrm{soc}(W_2) \cong 18b$ since $(18b)^\vee \cong 18c$ by **3.6(iii)**. Hence the structures of $P(18b)$ and $P(18c)$ in **3.6(vi)** yield that

$$W_1 \; = \; \boxed{\begin{array}{c} 18b \\ 9b \; 9c \\ 18c \end{array}} \qquad \text{and} \qquad W_2 \; = \; \boxed{\begin{array}{c} 18c \\ 9a \; 9d \\ 18b \end{array}} \qquad \blacksquare$$

## 8. Proof of main results

**8.1.Notation.** We still keep the notation $F$, $j$, $B'$, $f'$ and $g'$, see **6.4**, **3.7** and **5.2–5.4**. Set $E = SD_{16}$, and let $P \rtimes E$ be the canonical semi-direct product such that $E$ acts on $P$ faithfully. Recall that $\mathrm{Aut}(P) \cong \mathrm{GL}_2(3)$ since $P = C_3 \times C_3$, and hence $SD_{16}$ is a Sylow 2-subgroup of $\mathrm{GL}_2(3)$.

**8.2.Lemma.** *The non-principal block algebra $A$ of* HN *and the principal block algebra $A'$ of* HS *are Puig equivalent.*

**Proof.** Let $j$ be the same as in **3.6(ii)**. Since $jBj \cong \mathcal{O}[P \rtimes E] = B'$ as interior $P$-algebras by **3.6(ii)**, we can identify $jBj$ and $B'$. Define a functor $F'$ : mod-$B \to$ mod-$B'$ via $F'(-) = - \otimes_B Bj$. By **3.6(ii)**, $F'$ induces a Puig equivalence (which is stronger than a Morita equivalence) between $B$ and $B'$. In the following we use the information on the structures of PIMs in $B$ and $B'$ described in **3.6(vi)** and **5.2(iii)**, respectively, without quoting these statements.

Then, first of all, we know that $F'(18a) = 2a$ by looking at the PIMs $P(18a)$ and $P(2a)$. Similarly, we know at least that $\{F'(9a), F'(9b), F'(9c), F'(9d)\} = \{1a = k_{H'}, 1b, 1c, 1d\}$. It follows from **5.4** that $1x \otimes 1x = 1a$ for any $x \in \{a, b, c, d\}$ since they are just in $\mathrm{Irr}(E)$. Hence a technique of self-Puig equivalence in [31, 2.8.Lemma] can be used just as in the proof of [31, 6.8.Lemma]. Namely, we can assume that $F'(9a) = 1a$. Hence, by comparing the second Loewy layers of $P(9a)$ and $P(1a)$, we get $F'(18b) = 2b$. Similarly, by looking at the third Loewy layers of $P(9a)$ and $P(1a)$, we have $F'(9b) = 1b$. If we look at the fourth Loewy layers of these PIMs, then we know $F'(18c) = 2c$. Thus, by looking at the second Loewy layers of $P(18c)$ and $P(2c)$, we know also that $F'(9d) = 1d$. These mean that $F'(9c) = 1c$. Namely, we can assume that

$$\begin{gathered} F'(9a) = 1a,\ F'(9b) = 1b,\ F'(9c) = 1c,\ F'(9d) = 1d, \\ F'(18a) = 2a,\ F'(18b) = 2b,\ F'(18c) = 2c. \end{gathered} \tag{22}$$

We know by **7.13** that Case(a) or Case(b) happens.

Assume, first, that **Case(b)** occurs. Then, by bunching up **2.2**, **7.1**, **7.5**, **7.6**, **7.13** and **5.7**, we get the diagram shown in Table 1.

First, all the three functors above are given by bimodules which are $p$-permutation modules over $\mathcal{O}[G_1 \times H_1]$ for corresponding two finite groups $G_1$ and $H_1$, which are $\Delta P$-projective, and also which induce a stable equivalence of Morita type at each step, if we indentify the source algebra $jBj$ as $\mathcal{O}[P \rtimes E]$.

Secondly, it has to be noted that all non-simple modules in the above diagram are uniquely determined (up to isomorphism) by just the diagrams given in the above boxes: This is clear for $F(S_1)$, $F(S_2)$, $F(S_1)$, $f'(k_{G'})$, $f'(154)$, and $f'(22)$ anyway, as well as for $F(S_4)$ and $f'(1253)$ by the structure of $P(18a)$ and $P(2a)$ given in **3.6(vi)** and **5.2(iii)**.

To tackle $F(S_6)$, the structure of $P(18a)$ specified in **3.6(vii)** shows that $P(18a)$ has a unique quotient with composition factors $9d + 2 \times 18a + 18b + 18c$. Moreover, $P(9d)$ has a unique quotient with composition factors $9d + 18a + 18b$. Since they both have a unique submodule with composition factors $18a + 18b$, the glueing to yield $F(S_6)$ also is uniquely defined, and thus $F(S_6)$ is uniquely determined by the diagram given. For $f'(748)$ we argue similarly using **5.2(iv)**.

We consider $F(S_7)$: Note first that for $P(18b)$ there is no Alperin diagram defined. By **3.6(vi)**, let $X$ be the unique quotient module of $P(18b)$ having radical and socle series $\boxed{\begin{array}{c} 18b \\ 9b\ 9c \end{array}}$. By the structure of $P(18b)$ given in **3.6(vi)** we have $[\Omega(X), 18a]^B = 1$, hence using **3.6(vii)** there is a homomorphism $\varphi \in \mathrm{Hom}_B(P(18a), \Omega(X))$ such that

$$\mathrm{Im}(\varphi) \ = \ \boxed{\begin{array}{ccccc} & & 18a & & \\ & 18c & & 18b & \\ 9a & 9d & 18a & & \\ & 18b & & & \end{array}}$$

TABLE 1. Case(b).

| mod-$A$ | $\xrightarrow{F}$ | mod-$B$ | $\xrightarrow{F'}$ | mod-$B'$ | $\xrightarrow{f'^{-1}}$ | mod-$A'$ |
|---|---|---|---|---|---|---|
| $S_1$ | $\mapsto$ | $\boxed{9a}$ | $\mapsto$ | $\boxed{1a}$ | $\mapsto$ | $k_{G'}$ |
| $S_2$ | $\mapsto$ | $\boxed{9b}$ | $\mapsto$ | $\boxed{1b}$ | $\mapsto$ | 154 |
| $S_3$ | $\mapsto$ | $\boxed{9c}$ | $\mapsto$ | $\boxed{1c}$ | $\mapsto$ | 22 |
| $S_4$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 18a & \\ 18b & & 18c \\ & 18a & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 2a & \\ 2b & & 2c \\ & 2a & \end{smallmatrix}}$ | $\mapsto$ | 1253 |
| $S_5$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 18c & \\ 9a & & 9d \\ & 18b & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 2c & \\ 1a & & 1d \\ & 2b & \end{smallmatrix}}$ | $\mapsto$ | 321 |
| $S_6$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & & 18a & & 9d \\ & 18c & & 18b & \\ 9d & & 18a & & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & & 2a & & 1d \\ & 2c & & 2b & \\ 1d & & 2a & & \end{smallmatrix}}$ | $\mapsto$ | 748 |
| $S_7$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 18b & \\ 9b & & 9c \\ & 18c & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 2b & \\ 1b & & 1c \\ & 2c & \end{smallmatrix}}$ | $\mapsto$ | 1176 |

This implies $\Omega(X)/\mathrm{Im}(\varphi) \cong 18c$. Since $18c$ occurs exactly twice as a composition factor of $\Omega(X)$, and also is a composition factor of $\mathrm{Im}(\varphi)$, we conclude that $[\Omega(X), 18c]^B = 1$, thus $\dim_k[\mathrm{Ext}^1_B(X, 18c)] = 1$. Hence a module having radical and socle series $\boxed{\begin{smallmatrix} 18b \\ 9b\ 9c \\ 18c \end{smallmatrix}}$ is uniquely defined. For $F(S_5)$, $f'(1176)$, and $f'(321)$ we argue similarly.

Then, it follows from **2.15** that $A$ and $A'$ are splendidly stable equivalent of Morita type, that is, $A$ and $A'$ are stable equivalent which is realized by an $\mathcal{O}[G \times G']$-bimodule which is a $p$-permutation module and $\Delta P$-projective. Hence, first of all, the stable equivalence actually gives a Morita equivalence by a result of Linckelmann [37, Theorem 2.1(iii)]. Then, if we look at the proof of [37, Theorem 2.1(iii)] which is actually given in [37, Remark 2.7], we know that the Morita equivalence between $A$ and $A'$ gives a bijection such as $S_5 \leftrightarrow 321$. Hence, we must have equalities between the corresponding Cartan invariants, namely, $c(S_5, S_5) = c(321, 321)$. However, we get that $c(S_5, S_5) = 3$ by **4.1**, and on the other hand, that $c(321, 321) = 2$ by **5.8**. This is a contradiction. Thus, **Case(b)** cannot happen.

This means that only **Case(a)** occurs, as is shown in Table 2. Then, again the same argument given above still works. Namely, we have a Morita equivalence between $A$ and $A'$, and

Table 2. Case(a).

| mod-$A$ | $\xrightarrow{F}$ | mod-$B$ | $\xrightarrow{F'}$ | mod-$B'$ | $\xrightarrow{f'^{-1}}$ | mod-$A'$ |
|---|---|---|---|---|---|---|
| $S_1$ | $\mapsto$ | $\boxed{9a}$ | $\mapsto$ | $\boxed{1a}$ | $\mapsto$ | $k_{G'}$ |
| $S_2$ | $\mapsto$ | $\boxed{9b}$ | $\mapsto$ | $\boxed{1b}$ | $\mapsto$ | 154 |
| $S_3$ | $\mapsto$ | $\boxed{9c}$ | $\mapsto$ | $\boxed{1c}$ | $\mapsto$ | 22 |
| $S_4$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 18a & \\ 18b & & 18c \\ & 18a & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 2a & \\ 2b & & 2c \\ & 2a & \end{smallmatrix}}$ | $\mapsto$ | 1253 |
| $S_5$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 18b & \\ 9b & & 9c \\ & 18c & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 2b & \\ 1b & & 1c \\ & 2c & \end{smallmatrix}}$ | $\mapsto$ | 1176 |
| $S_6$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & & 18a & & 9d \\ & 18c & & 18b & \\ 9d & & 18a & & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & & 2a & & 1d \\ & 2c & & 2b & \\ 1d & & 2a & & \end{smallmatrix}}$ | $\mapsto$ | 748 |
| $S_7$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 18c & \\ 9a & & 9d \\ & 18b & \end{smallmatrix}}$ | $\mapsto$ | $\boxed{\begin{smallmatrix} & 2c & \\ 1a & & 1d \\ & 2b & \end{smallmatrix}}$ | $\mapsto$ | 321 |

hence the Morita equivalence is a Puig equivalence by a result of Puig (and, independently, of Scott) [55, Remark 7.5], see [40, Theorem 4.1]. ■

**8.3.Proofs of 1.3 and 1.4**. Recall that a Puig equivalence lifts from $k$ to $\mathcal{O}$ by a result of Puig [53, 7.8.Lemma] (see [62, (38.8)Proposition], and that so does a splendid Rickard equivalence by a result of Rickard [57, Theorem 5.2], see [15, P.75, lines $-17 \sim -16$]. Thus, it is enough to consider blocks $A$, $B$, $A'$ and $B'$ only over $k$. Thus, we get **1.4** by **8.2**.

By results of Okuyama [51, Example 4.8] and [52, Corollary 2], the conjectures **1.1** and **1.2** hold for $A'$. Namely, we get the following diagram:

$$\begin{tikzcd} A \arrow[r, "\text{Puig equiv.}"] & A' \arrow[d, "\text{splendid Rickard equiv.}"] \\ B & B' \arrow[l, "\text{Puig equiv.}"'] \end{tikzcd}$$

Therefore, we finally get that $A$ and $B$ are splendidly Rickard equivalent. That is, the proof of **1.3** is completed. ■

**8.4.Proof of 1.5**. We get **1.5** from **3.2** and **1.3**. ■

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