A NOTE ON APPLICATIONS OF THE 'VECTOR ENUMERATOR' ALGORITHM

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1. Introduction

It is one of the aims of computational representation theory to provide tools for the explicit construction of representations of abstractly given algebras. If the algebra under consideration is given by a finite algebra presentation and the representation searched for is given by a finite module presentation, then the algorithmic tool at hand is the so-called VectorEnumerator algorithm, see Section 1.1 and Theorem 1.2. In this note we are going to show how it is possible to exploit certain abstract situations to obtain algebra or module presentations, making them accessible for analysis using the VectorEnumerator algorithm.

In Section 2 we show how module presentations can be obtained in two particular situations. Firstly an explicit matrix representation can be used to give a module presentation, and secondly we consider so-called *local* modules. In Section 3 we show how presentations of induced modules and of certain epimorphic images are found. In Section 4 we are concerned with field extensions and how these affect algebra presentations.

In the final Section 5 we show by example, a so-called *Iwahori-Hecke algebra* over a certain cyclotomic field, how the techniques described in this note work together in practice. Iwahori-Hecke algebras have gained considerable interest in the modular representation theory of finite groups of Lie type. In particular for the exceptional types methods of computational representation theory have been of great help in understanding the representation theory of Iwahori-Hecke algebras. In fact, examples of this kind have been the original motivation for the present work. The example presented here is part of a broader examination of exceptional Iwahori-Hecke algebras done by the author in [11].

Recently, there is rising interest in certain generalizations of Iwahori-Hecke algebras, the so-called *cyclotomic Hecke algebras*, see e.g. [2]. Again there are exceptional types, which for the time being defy deeper theoretical analysis. The computational techniques described in this note again turn out to be very helpful to understand the structure and the representation theory of yclotomic Hecke algebras. At the moment this is work under progress and details on this will appear elsewhere.

Let us now begin by setting the stage for the main actor, where as a general reference see [1, Section III.2.8.].

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1.1. Let K be a field. Let $\mathfrak{X}$ be a finite set and $A_K(\mathfrak{X})$ the free unitary associative K-algebra over $\mathfrak{X}$. Let $R_K(\mathfrak{X}) \subseteq A_K(\mathfrak{X})$ be a finite subset of *algebra relators* and $\langle R_K(\mathfrak{X})\rangle \trianglelefteq A_K(\mathfrak{X})$ the ideal generated by $R_K(\mathfrak{X})$. Then the K-algebra $\langle \mathfrak{X}|R_K(\mathfrak{X})\rangle := A_K(\mathfrak{X})/\langle R_K(\mathfrak{X})\rangle$ is called a *finitely presented K-algebra.* Let $\pi_R \colon A_K(\mathfrak{X}) \to \langle \mathfrak{X}|R_K(\mathfrak{X})\rangle$ denote the natural K-algebra epimorphism.

Let A be a K-algebra, $\mathfrak{Y}$ a finite set and $M_A(\mathfrak{Y})$ the free right unital A-module over $\mathfrak{Y}$. Let $r_A(\mathfrak{Y}) \subseteq M_A(\mathfrak{Y})$ be a finite subset of *module relators* and $\langle r_A(\mathfrak{Y})\rangle \leq M_A(\mathfrak{Y})$ the submodule generated by $r_A(\mathfrak{Y})$. Then the A-module $\langle \mathfrak{Y}|r_A(\mathfrak{Y})\rangle := M_A(\mathfrak{Y})/\langle r_A(\mathfrak{Y})\rangle$ is called a *finitely presented A-module.* Finally, let $\pi_r \colon M_A(\mathfrak{Y}) \to \langle \mathfrak{Y}|r_A(\mathfrak{Y})\rangle$ denote the natural K-algebra epimorphism.

1.2. **Theorem.** There is an algorithm, the VectorEnumerator algorithm, which, for given $A := \langle \mathfrak{X}|R_K(\mathfrak{X})\rangle$ and $M := \langle \mathfrak{Y}|r_A(\mathfrak{Y})\rangle$ as in Section 1.1, terminates, if and only if $\dim_K(M)$ is finite. In this case, a K-basis $\mathfrak{B}$ of M and representing matrices, with respect to $\mathfrak{B}$, for the action of $\pi_R(\mathfrak{X})$ on M are returned.

Theorem 1.2 has been proven in [6, 7] and independently in [5].

For the time being there are the following implementations of the VectorEnumerator algorithm: for K a small finite prime field or the rationals [8]; an experimental version [9] in the computer algebra system GAP [14]; and for K a rational function field over the rationals [12], based on the implementation [8] and the FACTORY library [4] for arithmetic of multivariate polynomials.

2. Module presentations

2.1. **Matrix representations.** Let A be a K-algebra, which is as a K-algebra is generated by the finite set $\mathfrak{A} := \{a_1, \dots, a_s\}$ and $s := |\mathfrak{A}|$. Let M be an A-module such that $n := \dim_K(M)$ is finite and $M = \langle m_1, \dots, m_r\rangle$ as A-module for some $r \in \mathbb{N}$. Let $\mathfrak{B} := \{b_1, \dots, b_n\}$ be a K-basis of M such that $b_i := m_{l(i)}w_i$, where $1 \leq l(i) \leq r$ and $w_i \in A$ for all $1 \leq i \leq n$. Let the action of $a_k \in \mathfrak{A}$ on M be given as $b_i a_k = \sum_{j=1}^{r} a_{ij}^k \cdot b_j$, where $a_{ij}^k \in K$.

Note that a special case is given by $b_i = m_i$ and $w_i = 1$ for all $1 \leq i \leq n = r$. This means that representing matrices, with respect to $\mathfrak{B}$, for the action of $\mathfrak{A}$ on M are known. If only an A-module generating set of M is known, a K-basis $\mathfrak{B}$ can be found using variants of the *standard basis algorithm*, see [13].

2.2. **Theorem.** Let $\mathfrak{Y} := \{\mathfrak{Y}_1, \dots, \mathfrak{Y}_r\}$ and

$$r_A(\mathfrak{Y}) := \{\mathfrak{Y}_{l(i)} w_i a_k - \sum_{j=1}^{r} a_{ij}^k \cdot \mathfrak{Y}_{l(j)} w_j ; 1 \leq i \leq n, 1 \leq k \leq s\}.$$

Then $\alpha \colon \langle \mathfrak{Y}|r_A(\mathfrak{Y})\rangle \to M \colon \pi_r(\mathfrak{Y}_i) \mapsto m_i \subseteq M_A(\mathfrak{Y})$ is an A-module isomorphism.

Proof. There is an A-module homomorphism $M_A(\mathfrak{Y}) \to M \colon \mathfrak{Y}_i \mapsto m_i$, which has $r_A(\mathfrak{Y})$ in its kernel, and is an epimorphism, as $M = \langle m_1, \dots, m_r\rangle$ as A-module. As $\{\pi_r(\mathfrak{Y}_{l(i)})w_i ; 1 \leq i \leq n\}$ generates $\langle \mathfrak{Y}|r_A(\mathfrak{Y})\rangle$ as a K-vector space, we have $\dim_K(\langle \mathfrak{Y}|r_A(\mathfrak{Y})\rangle) \leq n$, and the assertion follows. $\sharp$

2.3. **Local modules.** Let $\dim_K(A)$ be finite. Let S be an irreducible A-module and M a finitely generated A-module. M is called S-*local*, if $M/\mathrm{rad}(M) \cong S$ as A-modules. For $a \in A$ let $a_M \in \mathrm{End}_K(M)$ denote the linear map induced by the action of a on M. An element $a \in A$ is called an S-*peakword*, if $\ker(a_{S'}) = \{0\}$ for all simple A-modules $S' \ncong S$ and $\dim_K(\ker(a_S^2)) = \dim_K(\mathrm{End}_A(S))$; for more details see [10]. If M is S-local and $a \in A$ is an S-peakword, then, by [10, Theorem 2.3], there exists $v \in M$ such that $va^t = 0$ for some $t \in \mathbb{N}$ and $M = \langle v \rangle$ as A-module. Conversely, if $M = \langle v \rangle$ as A-module, such that $va^t = 0$ for an S-peakword $a \in A$ and some $t \in \mathbb{N}$, then M is S-local. Hence the finitely presented A-module $M_{a,t} := \langle Y | Ya^t \rangle$, being generated by the element $\pi_{a,t}(Y) \in M_{a,t}$, is S-local and thus an epimorphic image of the projective cover P_S of S. As the identity on $M_A(Y)$ induces an A-module epimorphism $\alpha_{a,t+1} \colon M_{a,t+1} \to M_{a,t} \colon \pi_{a,t+1}(Y) \mapsto \pi_{a,t}(Y)$ for all $t \in \mathbb{N}$, we have $\dim_K(M_{a,t}) \leq \dim_K(M_{a,t+1}) \leq \dim_K(P_S) < \infty$.

2.4. **Theorem.** Let $t_0 := \min\{t \in \mathbb{N}; \dim_K(M_{a,t}) = \dim_K(M_{a,t+1})\}$, where $a \in A$ is an S-peakword. Then for all $t \in \mathbb{N}$ we have as A-modules

$$\langle Y | Ya^{t_0} \rangle = M_{a,t_0} \cong M_{a,t_0+t} \cong P_S.$$

Proof. By assumption α_{a,t_0+1} is an isomorphism. By induction we now assume $t \geq 2$. We have $\pi_{a,t_0+t}(Y)a \cdot a^{t_0+t-1} = 0$, hence the A-submodule $\langle \pi_{a,t_0+t}(Y)a \rangle \leq M_{a,t_0+t}$ is an epimorphic image of $M_{a,t_0+t-1} \cong M_{a,t_0+t-2}$. Hence we even have $0 = \pi_{a,t_0+t}(Y)a \cdot a^{t_0+t-2} = \pi_{a,t_0+t}(Y)a^{t_0+t-1}$. Thus $\beta_{a,t_0+t-1} \colon M_{a,t_0+t-1} \to M_{a,t_0+t} \colon \pi_{a,t_0+t-1}(Y) \mapsto \pi_{a,t_0+t}(Y)$ is well-defined and $\beta_{a,t_0+t-1} = \alpha^{-1}_{a,t_0+t}$. Finally, there exists $v \in P_S$ such that $va^s = 0$, for some $s \in \mathbb{N}$, and $P_S = \langle v \rangle$ as A-modules. Hence P_S is an epimorphic image of $M_{a,s}$. ♯

2.5. **Remarks.**
a) In certain situations there are efficient techniques to find S-peakwords, see [10]. Hence in these cases Theorem 2.4 provides a means to determine projective covers of simple A-modules.
b) Unfortunately, it is not always true that $M_{a,1}$ is isomorphic to the simple module S. In general, further relators are needed. These can e.g. be found by camparing the action of A on $M_{a,1}$ and on S, with respect to suitable standard K-bases of $M_{a,1}$ and S.

3. Induced modules and epimorphic images

3.1. **Induced modules.** Let A and B be K-algebras and $\varphi \colon A \to B$ a K-algebra homomorphism. Note that if φ is a monomorphism, then A can be considered as a K-subalgebra of B. Anyway, B can be considered as a left A-module ${}_AB$, by restriction along φ. For any A-module N we can form the *induced* B-module $N \otimes_A B$. Note that if ${}_AB$ is free of finite rank, s say, and $\dim_K(N)$ is finite, then we have $\dim_K(N \otimes_A B) = s \cdot \dim_K(N)$.

Let now $\mathfrak{Y}$ be a finite set and $r := |\mathfrak{Y}|$. Hence we have a B-module isomorphism $\alpha^r \colon M_A(\mathfrak{Y}) \otimes_A B \cong \oplus_{i=1}^r (A \otimes_A B) \to \oplus_{i=1}^r B \cong M_B(\mathfrak{Y})$, given componentwise by the natural isomorphism $\alpha \colon A \otimes_A B \to B \colon a \otimes b \mapsto \varphi(a)b$.

3.2. **Theorem.** Let $M := \langle \mathfrak{Y} | r_A(\mathfrak{Y}) \rangle$ be a finitely presented A-module, and let

$$r_B(\mathfrak{Y}) := \alpha^r(r_A(\mathfrak{Y}) \otimes 1) \subseteq M_B(\mathfrak{Y}).$$

Then we have $\langle \mathfrak{Y} | r_B(\mathfrak{Y}) \rangle \cong M \otimes_A B$ as B-modules.

Proof. By [15, Proposition 2.6.3], the tensor functor $? \otimes_A B$ is right exact. Hence we have $(M_A(\mathfrak{Y})/\langle r_A(\mathfrak{Y}) \rangle) \otimes_A B \cong (M_A(\mathfrak{Y}) \otimes_A B)/\langle r_A(\mathfrak{Y}) \otimes 1 \rangle$. ♯

3.3. **Epimorphic images.** If $M := \langle \mathfrak{Y} | r_A(\mathfrak{Y}) \rangle$ is a finitely presented A-module, and we add module relators to the set $r_A(\mathfrak{Y})$, we by definition obtain an epimorphic image of M. If $\langle \mathfrak{X} | R_K(\mathfrak{X}) \rangle$ is a finitely presented K-algebra, and we add algebra relators to the set $R_K(\mathfrak{X})$, we not only obtain an epimorphic image of the algebra we started with, but this also affects finitely presented modules.

Let again A and B be K-algebras and $\varphi \colon A \to B$ a K-algebra homomorphism. Note that in the above situation φ is an epimorphism. Anyway, any B-module N can be considered as an A-module N_A, by restriction along φ. In particular, we can consider B as an A-module B_A. We assume that $B_A = \langle \mathfrak{W} \rangle$ as A-module, where $\mathfrak{W} := \{w_1, \ldots, w_s\}$ and $s := |\mathfrak{W}|$. Note that $\mathfrak{W}$ can be chosen as a singleton set, if and only if φ is an epimorphism.

Let again $\mathfrak{Y}$ be a finite set and $r := |\mathfrak{Y}|$. There is an A-module epimorphism

$$\beta_s^r \colon \oplus_{j=1}^s M_A(\mathfrak{Y}) \cong \oplus_{j=1}^s (\oplus_{i=1}^r A) \cong \oplus_{i=1}^r (\oplus_{j=1}^s A) \to \oplus_{i=1}^r B_A \cong M_B(\mathfrak{Y})_A$$

given componentwise, i.e. for all $1 \leq i \leq r$, by $\beta_s \colon \oplus_{j=1}^s A \to B_A \colon [a_1, \ldots, a_s] \mapsto \sum_{j=1}^s w_j a_j$.

From this we conclude the following theorem, where for a subset $N \subseteq M_A(\mathfrak{Y})$ we let $[N, \ldots, N] := \{[n_1, \ldots, n_s] \in \oplus_{j=1}^s M_A(\mathfrak{Y}); n_j \in N, 1 \leq j \leq s\}$.

3.4. **Theorem.** Let $M := \langle \mathfrak{Y} | r_A(\mathfrak{Y}) \rangle$ be a finitely presented A-module and

$$r_{B,A}(\mathfrak{Y}) := \beta_s^r([r_A(\mathfrak{Y}), \ldots, r_A(\mathfrak{Y})]) \subseteq M_B(\mathfrak{Y}).$$

Then β_s^r induces an A-module epimorphism $\oplus_{j=1}^s M \to \langle \mathfrak{Y} | r_{B,A}(\mathfrak{Y}) \rangle_A$. In particular, if φ is an epimorphism, then $\langle \mathfrak{Y} | r_{B,A}(\mathfrak{Y}) \rangle_A$ is an epimorphic image of the A-module M.

4. Finitely generated field extensions

Besides the theoretical interest in rationality questions, this section is motivated by the following practical problem. To actually run the VectorEnumerator algorithm, one has to be able to do arithmetical operations in the base field, which for the interesting applications have to be sufficiently fast as well. In the case where an efficient implementation of the arithmetic in a field K is available, but the algebra and module presentations are given over a finite extension field $L \geq K$, for which such an implementation is not at hand, one can use the VectorEnumerator algorithm instead to do the arithmetic in L.

4.1. Let $L := K(\zeta) \geq K$ be an algebraic field extension. Note that this covers finitely generated separable field extensions, as in these cases there always exist primitive elements. Arbitrary finitely generated algebraic field extension can be dealt with by an iterated application of Proposition 4.2 and Theorem 4.3.

Let $\mu_K(X) \in K[X]$ the minimum polynomial of ζ over K. Hence $K[X]$ is isomorphic to the free K-algebra $A_K(X)$ being generated by the element $X \in A_K(X)$ and $\lambda\colon \langle X|\mu_K(X)\rangle \to L\colon \pi_\mu(X) \mapsto \zeta$ is a K-algebra isomorphism.

4.2. **Proposition.** Let $S_K(\mathfrak{X}, X) := \{\mu_K(X)\} \cup \{xX - Xx; x \in \mathfrak{X}\} \subseteq A_K(\mathfrak{X}, X)$, where $\mathfrak{X}$ is a finite set not containing X. Then the following holds.
a) There is a K-algebra isomorphism

$$\alpha\colon A_K(\mathfrak{X}) \otimes_K \langle X|\mu_K(X)\rangle \to \langle \mathfrak{X}, X|S_K(\mathfrak{X}, X)\rangle\colon x \otimes \pi_\mu(X) \mapsto \pi_S(xX)$$

for all $x \in \mathfrak{X}$. Hence $\langle \mathfrak{X}, X|S_K(\mathfrak{X}, X)\rangle$ becomes an L-algebra via $\alpha(1 \otimes \lambda^{-1})$.
b) There is an L-algebra isomorphism

$$\beta\colon \langle \mathfrak{X}, X|S_K(\mathfrak{X}, X)\rangle \to A_L(\mathfrak{X})\colon \begin{cases} \pi_S(X) & \mapsto \quad \zeta \cdot 1, \\ \pi_S(x) & \mapsto \quad x, \quad x \in \mathfrak{X}. \end{cases}$$

Proof. a) The monomorphisms $A_K(\mathfrak{X}) \to A_K(\mathfrak{X}, X)$ and $A_K(X) \to A_K(\mathfrak{X}, X)$ induce a K-linear map $A_K(\mathfrak{X}) \otimes_K A_K(X) \to A_K(\mathfrak{X}, X) \stackrel{\pi_S}{\to} \langle \mathfrak{X}, X|S_K(\mathfrak{X}, X)\rangle$, where $1 \otimes X \mapsto \pi_S(X)$ and $x \otimes 1 \mapsto \pi_S(x)$ for all $x \in \mathfrak{X}$. As $xX - Xx \in S_K(\mathfrak{X}, X)$ for all $x \in \mathfrak{X}$, this is a K-algebra homomorphism, having $A_K(\mathfrak{X}) \otimes_K \mu_K(X)$ in its kernel. Hence α is well-defined. Conversely, there is a K-algebra homomorphism $A_K(\mathfrak{X}, X) \to A_K(\mathfrak{X}) \otimes_K \langle X|\mu_K(X)\rangle$ defined by $X \mapsto 1 \otimes \pi_\mu(X)$ and $x \mapsto x \otimes 1$ for all $x \in \mathfrak{X}$, which hence has $S_K(\mathfrak{X}, X)$ in its kernel. Thus for the induced map $\tilde{\alpha}\colon \langle \mathfrak{X}, X|S_K(\mathfrak{X}, X)\rangle \to A_K(\mathfrak{X}) \otimes_K \langle X|\mu_K(X)\rangle$ we have $\tilde{\alpha} = \alpha^{-1}$.
b) There is a K-algebra homomorphism $A_K(\mathfrak{X}, X) \to A_L(\mathfrak{X})$ defined by $X \mapsto \zeta \cdot 1$ and $x \mapsto x$ for all $x \in \mathfrak{X}$, which has $S_K(\mathfrak{X}, X)$ in its kernel. Hence β is well-defined. There is an L-algebra homomorphism $\tilde{\beta}\colon A_L(\mathfrak{X}) \to \langle \mathfrak{X}, X|S_K(\mathfrak{X}, X)\rangle$ defined by $x \mapsto \pi_S(x)$ for all $x \in \mathfrak{X}$. By L-linearity, we have $\tilde{\beta}(\zeta) = \pi_S(X)$. Hence $\tilde{\beta} = \beta^{-1}$. $\sharp$

4.3. **Theorem.** Let $\langle \mathfrak{X}|R_L(\mathfrak{X})\rangle$ be a finitely presented L-algebra. For each $v \in R_L(\mathfrak{X}) \subseteq A_L(\mathfrak{X})$ choose $w_v \in \pi_S^{-1}\beta^{-1}(v) \subseteq A_K(\mathfrak{X}, X)$.
a) Let $R_K(\mathfrak{X}, X) := S_K(\mathfrak{X}, X) \cup \{w_v; v \in R_L(\mathfrak{X})\} \subseteq A_K(\mathfrak{X}, X)$, hence there is the natural K-algebra epimorphism $\pi_R^S\colon \langle \mathfrak{X}, X|S_K(\mathfrak{X}, X)\rangle \to \langle \mathfrak{X}, X|R_K(\mathfrak{X}, X)\rangle$. Then $\langle \mathfrak{X}, X|R_K(\mathfrak{X}, X)\rangle$ becomes an L-algebra via $\pi_R^S{\cdot}\alpha(1{\otimes}\lambda^{-1})$, and there is an L-algebra isomorphism

$$\gamma\colon \langle \mathfrak{X}, X|R_K(\mathfrak{X}, X)\rangle \to \langle \mathfrak{X}|R_L(\mathfrak{X})\rangle\colon \begin{cases} \pi_R(X) & \mapsto \quad \zeta \cdot 1, \\ \pi_R(x) & \mapsto \quad \pi_{R_L}(x), \quad x \in \mathfrak{X}. \end{cases}$$

b) Let $\langle \mathfrak{X}|R_L(\mathfrak{X})\rangle$ be *defined over* K, i.e. w_v can be chosen as elements of $A_K(\mathfrak{X})$ for all $v \in R_L(\mathfrak{X})$, and $R_K(\mathfrak{X}) := \{w_v; v \in R_L(\mathfrak{X})\} \subseteq A_K(\mathfrak{X})$. Then there is an L-algebra isomorphism

$$\delta\colon \langle \mathfrak{X}|R_K(\mathfrak{X})\rangle \otimes_K \langle X|\mu_K(X)\rangle \to \langle \mathfrak{X}|R_L(\mathfrak{X})\rangle\colon \pi_R(x) \otimes \pi_\mu(X) \mapsto \zeta \cdot \pi_{R_L}(x).$$

Proof. a) We have $\beta\pi_S(R_K(\mathfrak{X}, X)) = R_L(\mathfrak{X}) \cup \{0\}$. Hence γ is well-defined by $\gamma\pi_R^S = \pi_{R_L}\beta$ and the assertion follows from Proposition 4.2.
b) Let $R_K(\mathfrak{X}) \otimes 1 := \{w_v \otimes 1; v \in R_L(\mathfrak{X})\}$. The natural map

$$\langle \mathfrak{X} | R_K(\mathfrak{X}) \rangle \otimes_K \langle X | \mu_K(X) \rangle \to (A_K(\mathfrak{X}) \otimes_K \langle X | \mu_K(X) \rangle)/\langle R_K(\mathfrak{X}) \otimes 1 \rangle$$

is an L-algebra isomorphism. Let $\tilde{R}_K(\mathfrak{X}, X) := S_K(\mathfrak{X}, X) \cup R_K(\mathfrak{X})$. Hence we have $\alpha(R_K(\mathfrak{X}) \otimes 1) = \pi_S(\tilde{R}_K(\mathfrak{X}, X))$. By Proposition 4.2 and a) we conclude that δ is well-defined by $\delta \cdot (\pi_R^{-1} \otimes \mathrm{id}) = \gamma\pi_{\tilde{R}}^S \cdot \alpha$ and as asserted. ♯

4.4. **Corollary.** Let $A := \langle \mathfrak{X} | R_L(\mathfrak{X}) \rangle$. Let $M := \langle \mathfrak{Y} | r_A(\mathfrak{Y}) \rangle$ be a finitely presented A-module and $r := |\mathfrak{Y}|$.

a) Let $B := \langle \mathfrak{X}, X | R_K(\mathfrak{X}, X) \rangle$. By restriction along γ, any A-module N can be considered as a B-module, N_B say. In particular, there is the B-module isomorphism $\gamma^r \colon M_B(\mathfrak{Y}) \cong \oplus_{i=1}^r B \to \oplus_{i=1}^r A_B \cong M_A(\mathfrak{Y})_B$, which is given componentwise by γ. Let $r_B(\mathfrak{Y}) := (\gamma^r)^{-1}(r_A(\mathfrak{Y}))$. Then we have $\langle \mathfrak{Y} | r_B(\mathfrak{Y}) \rangle \cong M_B$ as B-modules.

b) Let A be defined over K and now $B := \langle \mathfrak{X} | R_K(\mathfrak{X}) \rangle$. Recall that $L \cong \langle X | \mu_K(X) \rangle$ as K-algebras. By restriction along δ, any A-module N can be considered as a $(B \otimes_K L)$-module, N_{BL} say, and there is the $(B \otimes_K L)$-module isomorphism $\delta^r \colon M_{B \otimes_K L}(\mathfrak{Y}) \cong \oplus_{i=1}^r (B \otimes_K L) \to \oplus_{i=1}^r A_{BL} \cong M_A(\mathfrak{Y})_{BL}$, which is given componentwise by δ. Let M *be defined over* K, i.e. for all $m \in r_A(\mathfrak{Y})$ we have $(\delta^r)^{-1}(m) = [m_1 \otimes 1, \ldots, m_r \otimes 1] \in \oplus_{i=1}^r (B \otimes_K 1)$.
Let $r_B(\mathfrak{Y}) := \{[m_1, \ldots, m_r]; m \in r_A(\mathfrak{Y})\}$. Then we have $\langle \mathfrak{Y} | r_B(\mathfrak{Y}) \rangle \otimes_K L \cong M_{BL}$ as $(B \otimes_K L)$-modules.

4.5. **Remark.** Let B and M_B be as in Corollary 4.4, assume that $\dim_K(M_B)$ is finite, and representing matrices, with respect to some K-basis $\mathfrak{B}$, for the action of $\pi_{R_K}(X)$ and $\pi_{R_K}(\mathfrak{X})$ on M_B are known. By Theorem 4.3, the action of $\pi_{R_K}(X)$ on M_B describes the action of the field generator $\zeta \in L$ on M. Hence using a variant of the standard basis algorithm, it is possible to find an L-basis $\tilde{\mathfrak{B}}$ for M and representing matrices, with respect to $\tilde{\mathfrak{B}}$, for the action of $\pi_{R_L}(\mathfrak{X})$ on M.

5. An example

The computations described in this section have been done using the implementation [8] of the VectorEnumerator algorithm, over the base field of rational numbers. On modern PCs the results are obtained in a matter of seconds or minutes of CPU time. Indeed, much larger examples can nowadays be dealt with using these techniques.

5.1. Let W be a finite Coxeter group of Dynkin type Γ with standard generators S. Let L be a field, $0 \neq u \in L$, and $H_L(\Gamma, u) := \langle \mathfrak{T} | R_L(\mathfrak{T}) \rangle$ be the the corresponding *Iwahori-Hecke algebra with parameter* u, where $\mathfrak{T} := \{T_s; s \in S\}$ and $R_L(\mathfrak{T})$ is defined as follows. For all $s, s' \in S$, $s \neq s'$, let $m_{ss'} \in \mathbb{N}$ be the order of $ss' \in W$. If $m_{ss'}$ is even, let

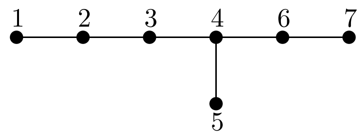
$$R(s, s') := (T_s T_{s'})^{m_{ss'}/2} - (T_{s'} T_s)^{m_{ss'}/2}.$$

If $m_{ss'}$ is odd, let $\lfloor \cdot \rfloor$ denote the lower Gauss bracket and

$$R(s, s') := (T_s T_{s'})^{\lfloor m_{ss'}/2 \rfloor} T_s - (T_{s'} T_s)^{\lfloor m_{ss'}/2 \rfloor} T_{s'}.$$

Let $R(s) := T_s^2 - (u-1)T_s - u \cdot 1$ and $R_L(\mathfrak{T}) := \{R(s), R(s, s'); s, s' \in S, s \neq s'\}$. For more details see e.g. [3, Section 10.].

TABLE 1. Dynkin graph of type E_7



5.2. From now on we let $\Gamma = E_7$, whose Dynkin graph is shown in Table 1, where $S := \{1, \ldots, 7\}$. This means $m_{ij} = 3$, if the nodes i and j are joined, and $m_{ij} = 2$, if they are not. We let $L := \mathbb{Q}[i] \geq \mathbb{Q} =: K$, hence we have $L \cong K[X]/\langle X^2 + 1\rangle$ as K-algebras. Finally, we let $u = -1$ and $H := H_L(E_7, -1)$. Note that H is defined over K, see Theorem 4.3. Hence we have a K-algebra $H_K := \gamma^{-1}\delta(H_K(E_7, -1) \otimes_K 1)$.

Let $H' \subseteq H$ be the *parabolic* L-subalgebra which as an L-algebra is generated by $\{T_2, \ldots, T_7\}$. We have $H' \cong H_L(E_6, -1)$ as L-algebras, where the Dynkin graph of type E_6 is the subgraph of the Dynkin graph of type E_7 induced by the nodes $\{2, \ldots, 7\}$. Let $H'_K := \gamma^{-1}\delta(H_K(E_6, -1) \otimes_K 1)$.

Let M the irreducible H'-module with character $\overline{\chi_{10,9}}$, see [3, Section 13.9.]. For the purpose of this note the precise definition of M does not matter, we only note that $\dim_L(M) = 10$. To construct M explicitly we use Theorem 1.2 and Corollary 4.4, which hence yields an H'_K-module $M_{H'_K}$ such that $\dim_K(M_{H'_K}) = 20$.

5.3. Let $H'' \subseteq H'$ be the parabolic L-subalgebra which as an L-algebra is generated by $\{T_3, \ldots, T_7\}$. We have $H'' \cong H_L(D_5, -1)$ as L-algebras, where the Dynkin graph of type D_5 is the subgraph of the Dynkin graph of type E_7 induced by the nodes $\{3, \ldots, 7\}$. Let $H''_K := \gamma^{-1}\delta(H_K(D_5, -1) \otimes_K 1)$.

It is known that the restriction $M_{H''}$ of M to H'' is an irreducible H''-*cell module*. Again, the precise definition of cell modules does not matter. The theory of cell modules allows us to compute representing matrices, with respect to a certain basis, for the action of $\{T_3, \ldots, T_7\}$ on $M_{H''}$. The cell module $M_{H''}$ is only defined over L, hence we use Theorem 2.2 and Corollary 4.4 to obtain a finitely presented H''_K-module $N := (M_{H''})_{H''_K}$, thus $\dim_K(N) = 20$.

Using Theorems 1.2 and 3.2, we compute the induced module $N \otimes_{H''_K} H'_K$. As H' is a free left H''-module of rank 27, which hence holds for the left H''_K-module H'_K as well, by Section 3.1 we have $\dim_K(N \otimes_{H''_K} H'_K) = 540$. By Frobenius reciprocity, see [15, Proposition 2.6.3], $M_{H'_K}$ is an epimorphic image of $N \otimes_{H''_K} H'_K$.

It turns out that the element $\pi_R(T_3T_5T_4T_6T_7) \in H''_K \subseteq H'_K$ acts on N by a K-linear map whose minimum polynomial equals $\Phi_4\Phi_8$, where $\Phi_n \in K[X]$ denotes the n-th cyclotomic polynomial. We hence add $\Phi_4\Phi_8(T_3T_5T_4T_6T_7)$ to the set of algebra relators in the presentation for H'_K, see Section 3.3. By Theorem 3.4, this defines an epimorphic image $\tilde{N}$ of $N \otimes_{H''_K} H'_K$, which in turn has $M_{H'_K}$ as an epimorphic image. Applying Theorem 1.2, we find that $\dim_K(\tilde{N}) = 20$, hence we have $\tilde{N} \cong M_{H'_K}$ as H'_K-modules.

5.4. Using the matrices for $M_{H'_K}$ and Theorems 1.2, 2.2, and 3.2 again, we compute the induced module $M_{H'_K} \otimes_{H'_K} H_K$. As H is a free left H'-module of rank 56, which hence holds for the left H'_K-module H_K as well, we have $\dim_K(M_{H'_K} \otimes_{H'_K} H_K) = 1120$.

To exhibit a certain epimorphic image of $M_{H'_K} \otimes_{H'_K} H_K$, we proceed as follows. Let $w_0 \in W$ be the longest element, where W is the Coxeter group of type E_7. The corresponding element $T_{w_0} \in H_K$, which is central in H_K, can be written as a product of length 63 in the generators $\{T_1, \ldots, T_7\}$ of H_K. We add $T_{w_0} + 1$ to the set of algebra relators in the presentation for H'_K, see Section 3.3, and now do the induction from H'_K to H_K by Theorem 3.2 and going over to the epimorphic image $\tilde{M}$ of $M_{H'_K} \otimes_{H'_K} H_K$ defined by the additional relator by Theorem 3.4 in a single application of Theorem 1.2. We find that $\dim_K(\tilde{M}) = 448$ holds.

Unfortunately, we cannot explain the significance of the existence of this epimorphic image in this note. For more details and for more examples of this kind, we refer the reader to [11].

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