Übungen zur Algebraischen Zahlentheorie (WS 2023)

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(9.1) Exercise: Conductors.
Let $K \subseteq L$ be an extension of algebraic number fields, and let $\mathfrak{a} \trianglelefteq \mathcal{O}_L$. Show that there is an integral primitive element α of L over K, such that $\mathfrak{a}$ and the conductor $\operatorname{ann}_{\mathcal{O}_L}(\mathcal{O}_L/\mathcal{O}_K[\alpha]) \trianglelefteq \mathcal{O}_L$ are coprime.

(9.2) Exercise: Ramification and discriminants.
Let $K := \mathbb{Q}(\omega)$ be an algebraic number field, where $\omega \in \mathcal{O} := \mathcal{O}_K$, and let p be a rational prime neither dividing the conductor $\operatorname{ann}_{\mathbb{Z}}(\mathcal{O}/\mathbb{Z}[\omega])$ nor the degree $[K\colon \mathbb{Q}]$. Show that p is ramified in K if and only if $p \mid \operatorname{disc}(\mathcal{O})$.

(9.3) Exercise: Discriminants of composite fields.
Let K and L be algebraic number fields, let $n := [K\colon \mathbb{Q}]$ and $m := [L\colon \mathbb{Q}]$, let $\mathcal{O} := \mathcal{O}_K$ and $\widetilde{\mathcal{O}} := \mathcal{O}_L$, and let $\delta := \operatorname{disc}(\mathcal{O})$ and $\widetilde{\delta} := \operatorname{disc}(\widetilde{\mathcal{O}})$. Let KL be the composite field KL, and let $\widehat{\mathcal{O}} := \mathcal{O}_{KL}$ be its ring of integers.

a) Show that if $[KL\colon \mathbb{Q}] = nm$, then $K \cap L = \mathbb{Q}$. Conversely, if K and L are Galois, show that $K \cap L = \mathbb{Q}$ implies $[KL\colon \mathbb{Q}] = nm$.

Now assume that KL has degree $[KL\colon \mathbb{Q}] = nm$.

b) Let $d \in \gcd(\delta, \widetilde{\delta})$. Show that $\mathcal{O}\widetilde{\mathcal{O}} \subseteq \widehat{\mathcal{O}} \subseteq \frac{1}{d} \cdot \mathcal{O}\widetilde{\mathcal{O}}$.
c) If δ and $\widetilde{\delta}$ are coprime, show that $\operatorname{disc}(\widehat{\mathcal{O}}) = \delta^m \cdot \widetilde{\delta}^n$.

(9.4) Exercise: Prime factorisation (partly GAP).
Let $\alpha \in \mathbb{R}$ such that $\alpha^5 = 5(\alpha + 1)$, let $K := \mathbb{Q}(\alpha)$, and let $\mathcal{O} := \mathcal{O}_K$.
a) Show that $\operatorname{disc}(\mathbb{Z}[\alpha]) = 3^2 \cdot 5^5 \cdot 41$, and that $\operatorname{ann}_{\mathbb{Z}}(\mathcal{O}/\mathbb{Z}[\alpha]) = (3) \trianglelefteq \mathbb{Z}$.
b) Using GAP, compute the factorisation of $p\mathcal{O} \triangleleft \mathcal{O}$ for the rational primes $p \neq 3$ up to 10^4 (say). What do you observe for the inertia degrees occurring?