Übungen zur Algebraischen Zahlentheorie (WS 2023)

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(2.1) Exercise: Pythagorean triples.
Show that the primitive positive integer solutions of the equation $X^2+Y^2=Z^2$ are the triples $[x,y,z]$ and $[y,x,z]$ such that $x=u^2-v^2$ and $y=2uv$ and $z=u^2+v^2$, for some $u,v\in\mathbb{Z}$ coprime such that $u>v\geq 1$ and $2\mid uv$.

Hint. Use the ring $\mathbb{Z}[i]$.

(2.2) Exercise: Quadratic number rings.
a) For $d\in\mathbb{Z}\setminus\{0,1\}$ square-free determine the ring of integers $\mathcal{O}_d$ of the **quadratic number field** $\mathbb{Q}(\sqrt{d})=\{a+b\sqrt{d}\in\mathbb{C};a,b\in\mathbb{Q}\}$. How does $\mathcal{O}_d$ relate to $\mathbb{Z}[\sqrt{d}]=\{a+b\sqrt{d}\in\mathbb{C};a,b\in\mathbb{Z}\}$? (Recall that $\mathcal{O}_{-1}=\mathbb{Z}[\sqrt{-1}]$.)
b) Show that there is $\mathbb{Q}$-basis $\mathcal{B}\subseteq\mathbb{Q}(\sqrt{d})$ which is contained in $\mathcal{O}_d$. Compute $\mathrm{disc}(\mathcal{B})$. (Try to choose $\mathcal{B}$ such that $|\mathrm{disc}(\mathcal{B})|$ is as small as possible.)

Hint. Distinguish the cases $d\equiv 1 \pmod 4$ and $d\not\equiv 1 \pmod 4$.

(2.3) Exercise: Units in quadratic number rings.
For $d\in\mathbb{Z}\setminus\{0,1\}$ square-free let $\mathcal{O}_d$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$.
a) For $d<0$ determine the group of units $\mathcal{O}_d^*$, and show that it is finite.
b) For $d>0$ show that $\mathcal{O}_d^*$ is infinite.
c) Show that $\mathcal{O}_2^*=\langle\pm 1\rangle\times\langle 1+\sqrt{2}\rangle\cong C_2\times C_\infty$.

Hint. Recall the norm map $N\colon\mathbb{Q}(\sqrt{d})\to\mathbb{Q}$.

(2.4) Exercise: Integral squares (partly GAP).
a) Show that amongst the sums $s_n:=1+2+\cdots+n=\frac{n(n+1)}{2}\in\mathbb{Z}$, where $n\in\mathbb{N}_0$, there are infinitely many squares.
b) Indeed, provide an (efficient) method to enumerate the integers n such that s_n is a square, together with s such that $s^2=s_n$, and implement it into GAP. How many $n\leq 10^{100}$ are there such that s_n is a square?

Hint. Use the ring $\mathbb{Z}[\sqrt{2}]$.