Übungen zur Algebraischen Zahlentheorie (WS 2023)

PD Dr. Jürgen Müller, **Ausgabe:** 21.12..2023

(11.1) Exercise: Quadratic and cyclotomic fields.
a) Show that any quadratic field is a subfield of a suitable cyclotomic field.
b) Let $n \geq 3$ be odd. Describe the quadratic subfields of $\mathbb{Q}(\zeta_n)$.
c) Let $k \geq 3$. Describe the quadratic subfields of $\mathbb{Q}(\zeta_{2^k})$.

(11.2) Exercise: Legendre symbols.
Let $p \in \mathcal{P}_\mathbb{Z}$ be odd. Show that p can be written as $p = a^2 + 2b^2$, where $a, b \in \mathbb{Z}$, if and only if $p \equiv 1 \pmod 8$ or $p \equiv 3 \pmod 8$.

(11.3) Exercise: Quadratic polynomials.
Let $a \in \mathbb{Z}$.
a) Show that any prime divisor p of $4a^2 + 1 \in \mathbb{Z}$ fulfills $p \equiv 1 \pmod 4$.
b) Show that any prime divisor p of $9a^2 + 3a + 1 \in \mathbb{Z}$ fulfills $p \equiv 1 \pmod 3$.

(11.4) Exercise: Values of polynomials.
Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be square-free, let $\mu_d := X^2 - d \in \mathbb{Z}[X]$, let $\mathcal{V}_d \subseteq \mathbb{Z}$ be the image of the associated polynomial function $\mu_d \colon \mathbb{Z} \to \mathbb{Z}$, and let $\mathcal{P}_d \subseteq \mathcal{P}_\mathbb{Z}$ be the set of prime divisors of the elements of $\mathcal{V}_d$.

Show that $\mathcal{P}_d = \mathcal{D}_d \,\dot\cup\, \mathcal{R}_d$, where $\mathcal{D}_d \subseteq \mathcal{P}_\mathbb{Z}$ is the set of prime divisors of $4d$, and $\mathcal{R}_d \subseteq \mathcal{P}_\mathbb{Z}$ is the preimage of a subgroup $\overline{\mathcal{R}}_d \leq (\mathbb{Z}/(4d))^*$ of index 2.