REPRESENTATIONS OF FINITE GROUPS OF LIE TYPE

LECTURE II: DELIGNE-LUSZTIG THEORY AND SOME APPLICATIONS

Gerhard Hiss

Lehrstuhl D für Mathematik
RWTH Aachen University

Summer School
Finite Simple Groups and Algebraic Groups
Berlin, August 31 – September 10, 2009
09/09/09

THREE CASES

AIM

Classify all irreducible representations of all finite simple groups and related finite groups.

In the following, let $G = \mathbf{G}^F$ be a finite reductive group of characteristic p and let k be an algebraically closed field.

It is natural to distinguish three cases:

1. $\mathrm{char}(k) = p$ (usually $k = \bar{\mathbb{F}}_p$); defining characteristic (cf. Jantzen's lectures)
2. $\mathrm{char}(k) = 0$; ordinary representations
3. $\mathrm{char}(k) > 0$, $\mathrm{char}(k) \neq p$; non-defining characteristic

Today I will talk about Case 2, so assume that $\mathrm{char}(k) = 0$ from now on.

A SIMPLIFICATION: CHARACTERS

Let V, V' be kG-modules.

The character afforded by V is the map

$$\chi_V : G \to k, \quad g \mapsto \mathrm{Trace}(g|V).$$

Characters are class functions.

V and V' are isomorphic, if and only if $\chi_V = \chi_{V'}$.

$\mathrm{Irr}(G) := \{\chi_V \mid V \text{ simple } kG\text{-module}\}$: irreducible characters

$\mathcal{C}$: set of representatives of the conjugacy classes of G

The **square** matrix

$$[\chi(g)]_{\chi \in \mathrm{Irr}(G), g \in \mathcal{C}}$$

is the ordinary character table of G.

AN EXAMPLE: THE ALTERNATING GROUP A_5

EXAMPLE (THE CHARACTER TABLE OF $A_5 \cong \mathrm{SL}_2(4)$)

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$$A = (1 - \sqrt{5})/2, \qquad *A = (1 + \sqrt{5})/2$$

$$1 \in 1a, \quad (1,2)(3,4) \in 2a, \quad (1,2,3) \in 3a,$$

$$(1,2,3,4,5) \in 5a, \quad (1,3,5,2,4) \in 5b$$

GOALS AND RESULTS

AIM

Describe all ordinary character tables of all finite simple groups and related finite groups.

Almost done:

1. For alternating groups: Frobenius, Schur
2. For groups of Lie type: Green, Deligne, **Lusztig**, Shoji, . . . (only "a few" character values missing)
3. For sporadic groups and other "small" groups:

Atlas of Finite Groups, Conway, Curtis, Norton, Parker, Wilson, 1986

THE GENERIC CHARACTER TABLE FOR $\mathrm{SL}_2(q)$, q EVEN

	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(m)$	$q+1$	1	$\zeta^{am}+\zeta^{-am}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi^{bn}-\xi^{-bn}$

$a, m = 1, \ldots, (q-2)/2, \qquad b, n = 1, \ldots, q/2,$

$\zeta := \exp(\frac{2\pi\sqrt{-1}}{q-1}), \qquad \xi := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

$\begin{bmatrix} \mu^a & 0 \\ 0 & \mu^{-a} \end{bmatrix} \in C_3(a)$ ($\mu \in \mathbb{F}_q$ a primitive $(q-1)$th root of 1)

$\begin{bmatrix} \nu^b & 0 \\ 0 & \nu^{-b} \end{bmatrix} \overset{\in}{\sim} C_4(b)$ ($\nu \in \mathbb{F}_{q^2}$ a primitive $(q+1)$th root of 1)

Specialising q to 4, gives the character table of $\mathrm{SL}_2(4) \cong A_5$.

DRINFELD'S EXAMPLE

The cuspidal simple $k\mathrm{SL}_2(q)$-modules have dimensions $q-1$ and $(q-1)/2$ (the latter only occur if p is odd).

How to construct these?

Consider the affine curve

$$C = \{(x, y) \in \bar{\mathbb{F}}_p^2 \mid xy^q - x^q y = 1\}.$$

$G = \mathrm{SL}_2(q)$ acts on C by linear change of coordinates.

Hence G also acts on the étale cohomology group

$$H_c^1(C, \bar{\mathbb{Q}}_\ell),$$

where ℓ is a prime different from p.

It turns out that the simple $\bar{\mathbb{Q}}_\ell G$-submodules of $H_c^1(C, \bar{\mathbb{Q}}_\ell)$ are the cuspidal ones (here $k = \bar{\mathbb{Q}}_\ell$).

DELIGNE-LUSZTIG VARIETIES

Let ℓ be a prime different from p and put $k := \bar{\mathbb{Q}}_\ell$.

Recall that $G = \mathbf{G}^F$ is a finite reductive group.

Deligne and Lusztig (1976) construct for each pair $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, a generalised character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of G.
(A generalised character of G is an element of $\mathbb{Z}[\mathrm{Irr}(G)]$.

Let $(\mathbf{T}, \theta)$ be a pair as above.

Choose a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$ with Levi subgroup $\mathbf{T}$.
(In general $\mathbf{B}$ is **not** F-stable.)

Consider the Deligne-Lusztig variety associated to $\mathbf{U}$,

$$Y_{\mathbf{U}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}.$$

This is an algebraic variety over $\bar{\mathbb{F}}_p$.

DELIGNE-LUSZTIG GENERALISED CHARACTERS

The finite groups $G = \mathbf{G}^F$ and $T = \mathbf{T}^F$ act on $Y_{\mathbf{U}}$, and these actions commute.

Thus the étale cohomology group $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)$ is a $\bar{\mathbb{Q}}_\ell G$-module-$\bar{\mathbb{Q}}_\ell T$,

and so its θ-isotypic component $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)_\theta$ is a $\bar{\mathbb{Q}}_\ell G$-module,

whose character is denoted by ch $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)_\theta$.

Only finitely many of the vector spaces $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)$ are $\neq 0$.

Now put

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \sum_i (-1)^i \mathrm{ch}\, H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)_\theta.$$

This is a Deligne-Lusztig generalised character.

PROPERTIES OF DELIGNE-LUSZTIG CHARACTERS

The above construction and the following facts are due to Deligne and Lusztig (1976).

FACTS

Let $(\mathbf{T}, \theta)$ be a pair as above. Then

1. $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *is independent of the choice of* $\mathbf{B}$ *containing* $\mathbf{T}$.
2. *If* θ *is in general position, i.e.* $N_G(\mathbf{T}, \theta)/T = \{1\}$, *then* $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *is an irreducible character.*

FACTS (CONTINUED)

3. *For* $\chi \in \mathrm{Irr}(G)$, *there is a pair* $(\mathbf{T}, \theta)$ *such that* χ *occurs in* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$.

A GENERALISATION

Instead of a torus $\mathbf{T}$ one can consider any F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$.

Warning: $\mathbf{L}$ does in general not give rise to a Levi subgroup of G as used in my first lecture.

Consider a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi complement $\mathbf{L}$ and unipotent radical $\mathbf{U}$, not necessarily F-stable.

The corresponding Deligne-Lusztig variety $Y_{\mathbf{U}}$ is defined as before: $Y_{\mathbf{U}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}$.

This is related to the one defined by Jean Michel:
$Y_{\mathbf{U}} \twoheadrightarrow \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g\mathbf{U} \cap F(g\mathbf{U}) \neq \emptyset\}, \quad g \mapsto g\mathbf{U}$.

One gets a Lusztig-induction map
$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : \mathbb{Z}[\mathrm{Irr}(L)] \to \mathbb{Z}[\mathrm{Irr}(G)], \mu \to R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\mu)$.

PROPERTIES OF LUSZTIG INDUCTION

The above construction and the following facts are due to Lusztig (1976).

Let **L** be an F-stable Levi subgroup of **G** contained in the parabolic subgroup **P**, and let $\mu \in \mathbb{Z}[\mathrm{Irr}(L)]$.

FACTS

1. $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\mu)(1) = \pm[G:L]_{p'} \cdot \mu(1)$.
2. *If* **P** *is F-stable, then $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\mu) = R^G_L(\mu)$ is the Harish-Chandra induced character.*
3. *Jean Michel's version of $Y_{\mathbf{U}}$ yields the same map $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$.*

It is not known, whether $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is independent of **P**, but it is conjectured that this is so.

UNIPOTENT CHARACTERS

DEFINITION (LUSZTIG)

A character χ of G is called unipotent, if χ is irreducible, and if χ occurs in $R_{\mathbf{T}}^{\mathbf{G}}(\mathbf{1})$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, where $\mathbf{1}$ denotes the trivial character of $T = \mathbf{T}^F$.

We write $\mathrm{Irr}^u(G)$ for the set of unipotent characters of G.

The above definition of unipotent characters uses étale cohomology groups.

So far, no elementary description known, except for $\mathrm{GL}_n(q)$; see below.

Lusztig classified $\mathrm{Irr}^u(G)$ in all cases, **independently** of q.

Harish-Chandra induction preserves unipotent characters (i.e. $\mathrm{Irr}^u(G)$ is a union of Harish-Chandra series), so it suffices to construct the **cuspidal** unipotent characters.

THE UNIPOTENT CHARACTERS OF $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$ and T the torus of diagonal matrices.

Then $\mathrm{Irr}^u(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi \text{ occurs in } R_T^G(\mathbf{1})\}$.

Moreover, there is bijection

$$\mathcal{P}_n \leftrightarrow \mathrm{Irr}^u(G), \quad \lambda \leftrightarrow \chi_\lambda,$$

where $\mathcal{P}_n$ denotes the set of partitions of n.

This bijection arises from $\mathrm{End}_{kG}(R_T^G(\mathbf{1})) \cong \mathcal{H}_{k,q}(S_n) \cong kS_n$.

The degrees of the unipotent characters are "polynomials in q":

$$\chi_\lambda(1) = q^{d(\lambda)} \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q-1)}{\prod_{h(\lambda)}(q^h - 1)},$$

with a certain $d(\lambda) \in \mathbb{N}$, and where $h(\lambda)$ runs through the hook lengths of λ.

Degrees of the unipotent characters of $\mathrm{GL}_5(q)$

λ	$\chi_\lambda(1)$
(5)	1
$(4,1)$	$q(q+1)(q^2+1)$
$(3,2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3,1^2)$	$q^3(q^2+1)(q^2+q+1)$
$(2^2,1)$	$q^4(q^4+q^3+q^2+q+1)$
$(2,1^3)$	$q^6(q+1)(q^2+1)$
(1^5)	q^{10}

JORDAN DECOMPOSITION OF CONJUGACY CLASSES

This is a model classification for $\mathrm{Irr}(G)$.

For $g \in G$ with Jordan decomposition $g = us = su$, we write $C^G_{u,s}$ for the G-conjugacy class containing g.

This gives a labelling

$$\{\text{conjugacy classes of } G\} \\ \updownarrow \\ \{C^G_{s,u} \mid s \text{ semisimple}, u \in C_G(s) \text{ unipotent}\}.$$

(In the above, the labels s and u have to be taken modulo conjugacy in G and $C_G(s)$, respectively.)

Moreover, $|C^G_{s,u}| = |G : C_G(s)||C^{C_G(s)}_{1,u}|$.

This is the Jordan decomposition of conjugacy classes.

EXAMPLE: THE GENERAL LINEAR GROUP ONCE MORE

$G = \mathrm{GL}_n(q)$, $s \in G$ semisimple. Then

$$C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$$

with $\sum_{i=1}^m n_i d_i = n$. (This gives finitely many class types.)

Thus it suffices to classify the set of unipotent conjugacy classes $\mathcal{U}$ of G.

By Linear Algebra we have

$$\mathcal{U} \longleftrightarrow \mathcal{P}_n = \{\text{partitions of } n\}$$

$$C^G_{1,u} \longleftrightarrow (\text{sizes of Jordan blocks of } u)$$

This classification is generic, i.e., independent of q.

In general, i.e. for other groups, it depends slightly on q.

JORDAN DECOMPOSITION OF CHARACTERS

Let $\mathbf{G}^*$ denote the reductive group dual to $\mathbf{G}$.
If $\mathbf{G}$ is determined by the root datum $(X, \Phi, X^\vee, \Phi^\vee)$, then $\mathbf{G}^*$ is defined by the root datum $(X^\vee, \Phi^\vee, X, \Phi)$.

EXAMPLES

(1) If $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$*, then* $\mathbf{G}^* = \mathbf{G}$*.*
(2) If $\mathbf{G} = \mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$*, then* $\mathbf{G}^* = \mathrm{Sp}_{2m}(\bar{\mathbb{F}}_p)$*.*

F gives rise to a Frobenius map on $\mathbf{G}^*$, also denoted by F.

MAIN THEOREM (LUSZTIG; JORDAN DEC. OF CHAR'S, 1984)

Suppose that $Z(\mathbf{G})$ *is connected. Then there is a bijection*

$$\mathrm{Irr}(G) \longleftrightarrow \{\chi_{s,\lambda} \mid s \in G^* \text{ semisimple }, \lambda \in \mathrm{Irr}^u(C_{G^*}(s))\}$$

(where the $s \in G^*$ *are taken modulo conjugacy in* G^**).*
Moreover, $\chi_{s,\lambda}(1) = |G^* : C_{G^*}(s)|_{p'}\, \lambda(1)$*.*

THE JORDAN DECOMPOSITION IN A SPECIAL CASE

Suppose that $s \in G^*$ is semisimple such that $\mathbf{L}^* := C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$.

This is the generic situation, e.g. it is always the case if $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ or if $|s|$ is divisible by good primes only.

Then there is an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, dual to $\mathbf{L}^*$.

By Lusztig's classification of unipotent characters, $\mathrm{Irr}^u(L)$ and $\mathrm{Irr}^u(L^*)$ can be identified.

Moreover, there is a linear character $\hat{s} \in \mathrm{Irr}(L)$, "dual" to $s \in Z(L^*)$, such that

$$\chi_{s,\lambda} = \pm R^{\mathbf{G}}_{\mathbf{L} \subset \mathbf{P}}(\hat{s}\lambda)$$

for all $\lambda \in \mathrm{Irr}^u(L) \leftrightarrow \mathrm{Irr}^u(L^*)$ (and some choice of $\mathbf{P}$).

THE IRREDUCIBLE CHARACTERS OF $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$. Then

$$\mathrm{Irr}(G) = \{\chi_{s,\lambda} \mid s \in G \text{ semisimple}, \lambda \in \mathrm{Irr}^u(C_G(s))\}.$$

We have $C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$ with $\sum_{i=1}^m n_i d_i = n$.

Thus $\lambda = \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_m$ with $\lambda_i \in \mathrm{Irr}^u(\mathrm{GL}_{n_i}(q^{d_i})) \longleftrightarrow \mathscr{P}_{n_i}$.

Moreover,

$$\chi_{s,\lambda}(1) = \frac{(q^n - 1) \cdots (q - 1)}{\prod_{i=1}^m \left[(q^{d_i n_i} - 1) \cdots (q^{d_i} - 1)\right]} \prod_{i=1}^m \lambda_i(1).$$

DEGREES OF THE IRREDUCIBLE CHARACTERS OF $\mathrm{GL}_3(q)$

$C_G(s)$	λ	$\chi_{s,\lambda}(1)$
$\mathrm{GL}_1(q^3)$	(1)	$(q-1)^2(q+1)$
$\mathrm{GL}_1(q^2)\times\mathrm{GL}_1(q)$	$(1)\boxtimes(1)$	$(q-1)(q^2+q+1)$
$\mathrm{GL}_1(q)^3$	$(1)\boxtimes(1)\boxtimes(1)$	$(q+1)(q^2+q+1)$
$\mathrm{GL}_2(q)\times\mathrm{GL}_1(q)$	$(2)\boxtimes(1)$ $(1,1)\boxtimes(1)$	q^2+q+1 $q(q^2+q+1)$
$\mathrm{GL}_3(q)$	(3) $(2,1)$ $(1,1,1)$	1 $q(q+1)$ q^3

(This example was already known to Steinberg.)

LUSZTIG SERIES

Lusztig (1988) also obtained a Jordan decomposition for $\mathrm{Irr}(G)$ in case $Z(\mathbf{G})$ is not connected, e.g. if $\mathbf{G} = \mathrm{SL}_n(\bar{\mathbb{F}}_p)$ or $\mathbf{G} = \mathrm{Sp}_{2m}(\bar{\mathbb{F}}_p)$ with p odd.

For such groups, $C_{\mathbf{G}^*}(s)$ is not always connected, and the problem is to define $\mathrm{Irr}^u(C_{G^*}(s))$, the unipotent characters.

The Jordan decomposition yields a partition

$$\mathrm{Irr}(G) = \bigcup_{(s) \subset G^*} \mathcal{E}(G, s),$$

where (s) runs through the semisimple G^*-conjugacy classes of G^* and $s \in (s)$.

By definition, $\mathcal{E}(G, s) = \{\chi_{s,\lambda} \mid \lambda \in \mathrm{Irr}^u(C_{G^*}(s)\}$.

For example $\mathcal{E}(G, 1) = \mathrm{Irr}^u(G)$.

The sets $\mathcal{E}(G, s)$ are called rational Lusztig series.

CONCLUDING REMARKS

1. The Jordan decompositions of conjugacy classes and characters allow for the construction of generic character tables in all cases.

2. Let $\{G(q) \mid q \text{ a prime power}\}$ be a series of finite groups of Lie type, e.g. $\{\mathrm{GU}_n(q)\}$ or $\{\mathrm{SL}_n(q)\}$ (n fixed, q variable).

   Then there exists a **finite** set $\mathcal{D}$ of polynomials in $\mathbb{Q}[x]$ s.t.:

   If $\chi \in \mathrm{Irr}(G(q))$, then there is $f \in \mathcal{D}$ with $\chi(1) = f(q)$.

BLOCKS OF FINITE GROUPS

Let G be a finite group and let $\mathcal{O}$ be a complete dvr of residue characteristic $\ell > 0$.

Then

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_r,$$

with indecomposable 2-sided ideals B_i, the blocks of $\mathcal{O}G$ (or ℓ-blocks of G).

Write

$$1 = e_1 + \cdots + e_r$$

with $e_i \in B_i$. Then the e_i are exactly the primitive idempotents in $Z(\mathcal{O}G)$ and $B_i = \mathcal{O}Ge_i = e_i\mathcal{O}G = e_i\mathcal{O}Ge_i$.

$\chi \in \mathrm{Irr}(G)$ belongs to B_i, if $\chi(e_i) \neq 0$.

This yields a partition of $\mathrm{Irr}(G)$ into ℓ-blocks.

A RESULT OF FONG AND SRINIVASAN

Let $G = \mathrm{GL}_n(q)$ or $U_n(q)$, where q is a power of p.

As for $\mathrm{GL}_n(q)$, the unipotent characters of $U_n(q)$ are labelled by partitions of n.

Let $\ell \neq p$ be a prime and put

$$e := \begin{cases} \min\{i \mid \ell \text{ divides } q^i - 1\}, & \text{if } G = \mathrm{GL}_n(q) \\ \min\{i \mid \ell \text{ divides } (-q)^i - 1\}, & \text{if } G = U_n(q). \end{cases}$$

(Thus e is the order of q, respectively $-q$ in $\mathbb{F}_\ell^*$.)

THEOREM (FONG-SRINIVASAN, 1982)

Two unipotent characters χ_λ, χ_μ of G are in the same ℓ-block of G, if and only if λ and μ have the same e-core.

Fong and Srinivasan found a similar combinatorial description for the ℓ-blocks of the other classical groups.

A RESULT OF BROUÉ AND MICHEL

Let again G be a finite reductive group of characteristic p and let ℓ be a prime, $\ell \neq p$.

For a semisimple ℓ'-element $s \in G^*$, define

$$\mathcal{E}_\ell(G, s) := \bigcup_{t \in C_{G^*}(s)_\ell} \mathcal{E}(G, st).$$

THEOREM (BROUÉ-MICHEL, 1989)

$\mathcal{E}_\ell(G, s)$ is a union of ℓ-blocks of G.

A RESULT OF CABANES AND ENGUEHARD

Let G and ℓ be as above.

Suppose $G = \mathbf{G}^F$ with $F(a_{ij}) = (a_{ij}^q)$ for some power q of p.

Write d for the order of q in $\mathbb{F}_\ell^*$.

A d-cuspidal pair is a pair $(\mathbf{L}, \zeta)$, where $\mathbf{L}$ is an F-stable d-split Levi subgroup of $\mathbf{G}$, and $\zeta \in \mathrm{Irr}(L)$ is d-cuspidal.

THEOREM (CABANES-ENGUEHARD, 1994)

(Some mild conditions apply.) Suppose that B is an ℓ-block of G contained in $\mathcal{E}_\ell(G, 1)$, the union of unipotent blocks.

Then there is a d-cuspidal pair $(\mathbf{L}, \zeta)$ such that

$$B \cap \mathcal{E}(G, 1) = \{\chi \in \mathrm{Irr}^u(G) \mid \chi \text{ is a constituent of } R^{\mathbf{G}}_{\mathbf{L} \subset \mathbf{P}}(\zeta)\}.$$

A similar description applies for $B \cap \mathcal{E}(G, t)$ with $t \in (G^*)_\ell$.

Thank you for your listening!