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SMALL DEGREE REPRESENTATIONS OF FINITE CHEVALLEY GROUPS IN DEFINING CHARACTERISTIC

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Abstract

We determine for all simple simply connected reductive linear algebraic groups defined over a finite field all irreducible representations in their defining characteristic of degree below some bound. These also give the small degree projective representations in defining characteristic for the corresponding finite simple groups. For large rank l our bound is proportional to l^3 and for rank $\leqslant 11$ much higher. The small rank cases are based on extensive computer calculations.

Note on this version: This is a revised preprint version of the published article [16], prepared in January 2016. The only differences are small updates of the references and a correction of Appendix 6.3 about Frobenius-Schur indicators (including the corresponding remark in the example 6.4). In the original article some indicators in types B_l with $l \equiv 0,3 \pmod 4$, and in types D_l with $l \equiv 0,1 \pmod 4$ were given as -1 while in those cases they are in fact all $+1$ (in those cases the element on which the highest weight must be evaluated to find the indicator is the trivial element and no involution). I thank John Bray and Bob Guralnick for making me aware of this error (and hints about what my mistake was).

We remark that the full characters of all representations whose degree is given in the tables of the Appendix 6 (and also of some others) can be found on the authors webpage.

1. *Introduction*

In this note we give lists of projective representations of simple Chevalley groups in their defining characteristic. There are two types of results.

First we determine for groups of rank $\leqslant 11$ *all* such representations of degree less than or equal some bound depending on the type (e.g., 100000 for type E_8), see Theorem 4.4 for details. These data are produced using a collection of computer programs developed by the author. Of course, some of the degrees given in our tables have been known before. To give two examples: Gilkey and Seitz handle some cases for exceptional types with computational methods in [7]. Some small rank cases are handled more theoretically like type B_2 and G_2 by Jantzen in [12] (although the explicit degrees are still not too easy to obtain from that result).

Secondly we determine for groups of classical type with rank l all representations of degree at most $l^3/8$ for type A_l, and of degree at most l^3 for the other types. For large l this is a small list, given in Theorem 5.1, and for small l this range is easily covered by our tables mentioned above. This extends similar results by Liebeck with bounds proportional to l^2, see [14, 5.4.11] and the references given there.

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Our results contains in particular a complement to the tables of representations in non-defining characteristic up to degree 250 worked out by Hiß and Malle in [8].

We fix some notation for the whole paper. Let $\tilde{G}$ be a finite twisted or non-twisted simple Chevalley group in characteristic p (we consider ${}^2F(2)'$ as sporadic and exclude it here, but see the corresponding remark after Theorem 4.4).

There is an associated connected reductive simple algebraic group G over $\bar{\mathbb{F}}_p$ of simply connected type, a Frobenius endomorphism F of G, a number $q = p^f$ with $2f \in \mathbb{Z}$ and an $m \in \mathbb{N}$ such that:
— G is defined over $\mathbb{F}_{q^m}$ via F^m.
— For the group of F-fixed points $G(q)$ with center Z we have $\tilde{G} \cong G(q)/Z$.
— $G(q)$ is the quotient of the universal covering group of $\tilde{G}$ by the p-part of its center.
(Note, that this includes the Suzuki and Ree groups.)

So, asking for the projective representations of $\tilde{G}$ in characteristic p is the same as asking for the representations of $G(q)$ in characteristic p. These can be constructed by restricting certain representations, called highest weight representations, of the algebraic group G to $G(q)$. This is explained in Section 2. In Section 3 we shortly describe how our computer programs for computing weight multiplicities work. In Section 4 we describe our main result consisting of lists of small degree representations for groups of rank at most 11. The lists are printed in Appendix 6. Finally, in Section 5 we consider groups of larger rank.

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2. *Representations in defining characteristic*

There are several well readable introductions to this topic, for example Humphreys' survey [10]. A detailed reference is Jantzen's book [13]. We recall some of the basic facts.

For G of simply connected type of rank l as in the introduction let T be a maximal torus of G, $X \cong \mathbb{Z}^l$ its character group and $Y \cong \mathbb{Z}^l$ its co-character group.

Let $\{\alpha_1, \ldots, \alpha_l\} \subset X$ be a set of simple roots for G with respect to T and $\alpha_i^\vee \in Y$ the coroot corresponding to α_i, $i = 1, \ldots, l$. Viewing $X \otimes \mathbb{R}$ as Euclidean space we define the fundamental weights $\omega_1, \ldots, \omega_l \in X \otimes \mathbb{R}$ as the dual basis of $\alpha_1^\vee, \ldots, \alpha_l^\vee$. This is a $\mathbb{Z}$-basis of X (because G is simply connected).

There is a partial ordering $\leqslant$ on X defined by $\omega \leqslant \omega'$ if and only if $\omega' - \omega$ is a non-negative linear combination of simple roots. A weight $\omega \in X$ is called dominant if it is a non-negative linear combination of the fundamental weights. The Weyl group W of G, generated by the reflections along the α_i, acts on X. Under this action each W-orbit on X contains a unique dominant weight.

From now on let L be a finite dimensional G-module over $\bar{\mathbb{F}}_p$. Considering this as T-module there is a direct sum decomposition $L = \bigoplus_{\omega \in X} L_\omega$ into weight spaces L_ω such that $t \in T$ acts by multiplication with $\omega(t)$ on L_ω. The set of $\omega \in X$ with $L_\omega \neq \{0\}$ is called the set of weights of L. The set of weights of L is a union of W-orbits and for $\omega \in X, x \in W$ we have $\dim(L_\omega) = \dim(L_{\omega x})$.

The following basic results characterize the irreducible representations L via their set of weights.

Theorem 2.1 (Chevalley)**.** *Let G and L be as above.*

(a) *If L is irreducible then the set of weights of L contains a (unique) element λ such that for all weights ω of L we have $\omega \leqslant \lambda$. This λ is called the* highest weight *of L, it is* *dominant and we have* $\dim(L_\lambda) = 1$.

(b) *An irreducible G-module L is determined up to isomorphism by its highest weight.*

(c) *For each dominant weight* $\lambda \in X$ *there is an irreducible G-module* $L(\lambda)$ *with highest weight* λ.

A dominant weight $\lambda = a_1\omega_1 + \cdots + a_l\omega_l \in X$ is called p-restricted if $0 \leqslant a_i \leqslant p-1$ for $1 \leqslant i \leqslant l$.

The following result of Steinberg shows how all highest weight modules $L(\lambda)$ of G can be constructed out of those with p-restricted highest weights.

Theorem 2.2 (Steinberg's tensor product theorem)**.** *Let* F_0 *be the Frobenius automorphism of* $\bar{\mathbb{F}}_p$*, raising elements to their p-th power. Twisting the G-action on a G-module L with* F_0^i, $i \in \mathbb{Z}_{\geqslant 0}$*, we get another G-module which we denote by* $L^{(i)}$*. If* $\lambda_0, \ldots, \lambda_n$ *are p-restricted weights then*

$$L(\lambda_0 + p\lambda_1 + \cdots + p^n\lambda_n) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \cdots \otimes L(\lambda_n)^{(n)}.$$

Finally we need to recall the relation between the irreducible modules of the algebraic group G and those of the finite group $G(q)$. This is nicely described by Steinberg in [21, 13.3, 11.6].

Theorem 2.3 (Steinberg)**.** *Let G and* $G(q)$ *be as in the introduction. We define a subset* Λ *of dominant weights. If* $G(q)$ *is not a Suzuki or Ree group (i.e., not of type* ${}^2B_2$, ${}^2G_2$ *or* ${}^2F_4$*) then* $\Lambda = \{a_1\omega_1 + \cdots + a_l\omega_l \mid 0 \leqslant a_i \leqslant q-1 \text{ for } 1 \leqslant i \leqslant l\}$*. In the case of Suzuki and Ree groups we define* $\Lambda = \{a_1\omega_1 + \cdots + a_l\omega_l \mid 0 \leqslant a_i \leqslant q/\sqrt{p} - 1$ *if* α_i *is a long root,* $0 \leqslant a_i \leqslant q\sqrt{p} - 1$ *if* α_i *is a short root*$\}$ *(note that q is the square root of an odd power of* $p = 2, 3$ *or* 2*, respectively, in these cases).*

Then the restrictions of the G-modules $L(\lambda)$ *with* $\lambda \in \Lambda$ *to* $G(q)$ *form a set of pairwise inequivalent representatives of all equivalence classes of irreducible* $\bar{\mathbb{F}}_p G(q)$*-modules.*

These results show that the dimensions of the irreducible representations of the groups $G(q)$ over $\bar{\mathbb{F}}_p$ are easy to obtain if we know the dimensions of the representations $L(\lambda)$ of the algebraic groups G for p-restricted weights λ.

3. *Computation of weight multiplicities*

In this section we sketch how we compute the degree of the representation $L(\lambda)$ for given root datum of G, highest weight λ and prime p. For almost all p it is the same as for the algebraic group over the complex numbers with same root datum, respectively its Lie algebra. In these cases the degree can be computed by a formula of Weyl, see [9, 24.3].

In the other cases no formula is known. But in principle there is an algorithm to compute the degree. This is described in [9, Exercise 2 of 26.4] and goes back to Burgoyne [3]. This was also used in [7] to handle some cases in exceptional groups. The idea is to construct a so-called Weyl module $V(\lambda)_\mathbb{Z}$ generically over the integers. By base change this leads to a module $V(\lambda)$ for any G with the given root datum over any ring, which has λ as a highest weight. Over $\mathbb{C}$ or over $\bar{\mathbb{F}}_p$ for almost all p this is irreducible and so isomorphic to $L(\lambda)$. In general $L(\lambda)$ is a quotient of $V(\lambda)$.

To construct $V(\lambda)_\mathbb{Z}$ one considers the universal enveloping algebra $\mathcal{U}$ of the complex Lie algebra corresponding to the given root datum. It contains a $\mathbb{Z}$-lattice $\mathcal{U}_\mathbb{Z}$, the Kostant $\mathbb{Z}$-form of $\mathcal{U}$, which is defined via a Chevalley basis of the Lie algebra. Up to equivalence there is a unique irreducible highest weight representation $V(\lambda)$ for $\mathcal{U}$ with highest weight λ. Let $0 \neq v \in V(\lambda)$ be a vector of weight λ (this is unique up to scalar). Then we set $V(\lambda)_{\mathbb{Z}} := \mathcal{U}_{\mathbb{Z}} v = \mathcal{U}^-_{\mathbb{Z}} v$.

We fix an ordering $\beta_1, \ldots, \beta_N$ of the set of positive roots. Then $\mathcal{U}^-_{\mathbb{Z}}$ and $\mathcal{U}^+_{\mathbb{Z}}$ have $\mathbb{Z}$-bases labeled by sequences $\mathbf{a} = (a_1, \ldots, a_N)$ of non-negative integers. Applying such a basis element $f_{\mathbf{a}}$ of $\mathcal{U}^-_{\mathbb{Z}}$ to a vector of weight ω we get a vector of weight $\omega - a_1\beta_1 - \cdots - a_N\beta_N$ (and similarly for basis vectors $e_{\mathbf{a}}$ of $\mathcal{U}^+_{\mathbb{Z}}$). So, decomposing $V(\lambda)_{\mathbb{Z}} = \bigoplus_{\omega \in X} (V_\omega)_{\mathbb{Z}}$ according to the weight spaces, we can describe generating sets of $(V_\omega)_{\mathbb{Z}}$ by all non-negative linear combinations of positive roots which are equal to $\lambda - \omega$.

There is a non-degenerate bilinear form $(\cdot,\cdot)$ on $V(\lambda)_{\mathbb{Z}}$ which can be described via this generating system. Different weight spaces are orthogonal with respect to this form. Let $\mathbf{a}, \mathbf{b}$ be two coefficient vectors as above such that the corresponding linear combination of the positive roots is the same. Then $e_{\mathbf{b}} f_{\mathbf{a}} v = n_{\mathbf{a},\mathbf{b}} v$ is again of weight λ and one defines $(f_{\mathbf{a}} v, f_{\mathbf{b}} v) := n_{\mathbf{a},\mathbf{b}} \in \mathbb{Z}$.

Since the form is nondegenerate we can compute the rank of a weight lattice $(V_\omega)_{\mathbb{Z}}$ by computing the rank of the matrix $(n_{\mathbf{a},\mathbf{b}})$ where $\mathbf{a}$ and $\mathbf{b}$ are running through all non-negative linear combinations of positive roots for $\lambda - \omega$. The dimension of the weight space L_ω of $L(\lambda)$ is the rank of $(n_{\mathbf{a},\mathbf{b}})$ modulo p.

The coefficients $n_{\mathbf{a},\mathbf{b}}$ can be computed by simplifying the element $e_{\mathbf{b}} f_{\mathbf{a}}$ with the help of the commutator relations in $\mathcal{U}$, see [9, 25.]. These involve the structure constants for a Chevalley basis of the corresponding Lie algebra which can be computed as described in [4, 4.2].

In principle this allows the determination of $\dim(L(\lambda))$ for all G, p and λ. But in practice these computations can become very long, already in small examples. There are technical problems like the question how to apply commutator relations most efficiently in order to compute the integers $n_{\mathbf{a},\mathbf{b}}$. Different strategies change the number of steps in the calculation considerably. But the main problem is that the generating sets for the weight spaces as described above are very redundant. Usually the number of non-negative linear combinations of positive roots which yield $\lambda - \omega$ is much larger than the dimension of V_ω. By a careful choice of the ordering of the positive roots one can reduce the linear combinations to consider, because for many $\mathbf{a} = (a_1, \ldots, a_N)$ with $\omega = \lambda - a_1\beta_1 - \cdots - a_N\beta_N$ there is a $1 \leqslant k \leqslant N$ such that $\lambda - a_1\beta_1 - \cdots - a_k\beta_k$ is not a weight of $V(\lambda)$ (and hence $f_{\mathbf{a}} v = 0$).

But the main improvement we get by using results of Jantzen and Andersen, see [13, II.8.19]. The so-called Jantzen sum formula expresses the determinant of the Gram matrix of the bilinear form $(\cdot,\cdot)$ on the lattice $(V_\omega)_{\mathbb{Z}}$ in terms of weight multiplicities of various $V(\mu)$ with $\mu \leqslant \lambda$. The weight multiplicities of these $V(\mu)$ can be efficiently computed by Freudenthal's formula, see [9, 22.3]. In particular the formula gives exactly the set of primes p for which the Weyl module $V(\lambda)$ is not isomorphic to $L(\lambda)$. In rare cases it happens that a prime p divides such a determinant exactly once - then we know without further calculations that $\dim(L_\omega) = \dim(V_\omega) - 1$.

Using these determinants we compute only parts of the matrices $(n_{\mathbf{a},\mathbf{b}})$ corresponding to a subset of its rows and columns until the submatrix has the full rank $\dim(V_\omega)$ and the product of its elementary divisors is equal to the known determinant. Then the rank of $(n_{\mathbf{a},\mathbf{b}})$ modulo p is the same as the rank of this submatrix modulo p.

With this approach we never need submatrices of $(n_{\mathbf{a},\mathbf{b}})$ of much larger dimension than $\dim(V_\omega)$. (In [7] the consideration of the matrices $(n_{\mathbf{a},\mathbf{b}})$ was substituted by computing somewhat smaller matrices using the action of parabolic subalgebras in cases where λ has a non-trivial stabilizer in W, but these matrices were still big compared to the dimension of the weight spaces.) Of course, our method is limited to representations where no single weight space has a dimension of more than a few thousand.

Actually we can also compute by our approach the Jantzen filtrations of the Weyl modules, see [13, II.8], by computing the elementary divisors of the matrices $(n_{\mathbf{a},\mathbf{b}})$ and not just their rank. One needs a sophisticated algorithm to compute the exact elementary divisors of such matrices of dimension bigger than 200, say. We developed the algorithm described in [15] for this purpose.

We have a collection of computer programs for doing the calculations described above; they are based on the computer algebra system GAP [19] and the package CHEVIE [6].

It is planned to make them available to other users in form of another package. This will be built on a new version of CHEVIE for GAP - Version 4 which is currently developed. While rewriting the current programs we hope to improve their efficiency and to extend their functionality. We postpone a much more detailed version of this very sketchy section until this package is ready.

4. *Representations of small degree for groups of small rank*

From Theorem 2.2 we see how to construct any highest weight representation $L(\lambda)$ of G from those with p-restricted weights by twisting with field automorphisms and tensoring.

Assume now we are given the type of an irreducible root system and a number $M \in \mathbb{N}$. We consider the groups G over $\bar{\mathbb{F}}_p$ with this root system for all p at once. We want to find for all primes p all p-restricted dominant weights λ such that the highest weight representation $L(\lambda)$ of the group G over $\bar{\mathbb{F}}_p$ has degree smaller or equal M.

Our main tool to restrict this question to a finite number of λ to consider is the following result by Premet, see [18]. Recall from Section 2 that the set of weights of $L(\lambda)$ is a union of W-orbits and that each W-orbit contains a unique representative which is dominant.

Theorem 4.1 (Premet)**.** *If the root system of G has different root lengths we assume that $p \neq 2$ and if G is of type G_2 we also assume $p \neq 3$. Let λ be a p-restricted dominant weight. Then the set of weights of $L(\lambda)$ is the union of the W-orbits of dominant weights ω with $\omega \leqslant \lambda$.*

We note that in the exceptional cases with different root lengths and $p = 2$, respectively $p = 3$, the statement of the theorem does actually not hold.

The following remark shows how to compute explicitly the number of weights of $L(\lambda)$ as given by Premet's theorem.

Remark 4.2. **(a)** Let $\omega = a_1\omega_1 + \cdots + a_l\omega_l$ be a dominant weight. Then the stabilizer of ω in the Weyl group W is the parabolic subgroup generated by the reflections along the simple roots α_i for which $a_i = 0$.

(b) For given dominant weight λ one finds the set $\Omega = \{\omega | \ \omega \text{ dominant and } \omega \leqslant \lambda\}$ efficiently as follows: Initialize $\Omega \leftarrow \{\lambda\}$. For each $\omega \in \Omega$ and for each positive root α compute $\omega - \alpha$. If this is also dominant add it to the set Ω.

Proof. For (a) see [9, 10.3B] and for (b) see [17, Prop.1]. □

Here is an algorithm for finding a set of candidate highest weights for representations of G of rank at most a given bound. It shows in particular that for any given bound there is only a finite number of highest weights which must be considered.

Algorithm 4.3. **Input:** An irreducible root system and an $M \in \mathbb{N}$.

Output: A set A of dominant weights which contains all λ such that there is a prime p for which λ is p-restricted and a group G over $\bar{\mathbb{F}}_p$ corresponding to the given root system with highest weight representation $L(\lambda)$ of degree at most M.

(1) For a dominant weight λ we define $m(\lambda)$ as the number of weights in all W-orbits of dominant weights ω with $\omega \leqslant \lambda$. The numbers $m(\lambda)$ can be computed using 4.2.

(2) We initialize the set A with all 2-restricted weights λ with $m(\lambda) \leqslant M$.

(3) We choose a linear function $\gamma : X \to \mathbb{Z}$ which takes positive values on the simple roots α_i, $1 \leqslant i \leqslant l$ (and hence on the fundamental weights).

Then we start to enumerate recursively for $n = 0, 1, \ldots$ all dominant weights λ with $\gamma(\lambda) = n$. (Clearly, for fixed n there is a finite number of such weights.) For each of these λ we compute $m(\lambda)$ and if it is $\leqslant M$ we put λ into A.

We proceed until we find an interval $I = [a, a + \max(\gamma(\alpha_i) \mid 1 \leqslant i \leqslant l)]$ such that $a > \gamma(\lambda)$ for all $\lambda \in A$ and such that for all dominant λ with $\gamma(\lambda) \in I$ we have $m(\lambda) > M$.

(4) If the given root system has roots of different length we add all 2-restricted weights to A. If there is a component of type G_2 we also add all 3-restricted weights.

(5) We return the set A.

Proof. We have to show that we will eventually find an interval I, as described in step (3), and that all dominant weights which were not considered during the algorithm lead to modules of dimension $> M$.

In the cases of root systems with different root lengths and $p = 2$ or $p = 3$ for which Premet's theorem 4.1 does not give a statement we have put in step (4) all 2-, respectively 3-restricted weights into A. So, from now on we only consider weights and primes for which Premet's theorem is applicable. This theorem says that $m(\lambda)$, defined in (1), is a lower bound for $\dim(L(\lambda))$ for all primes p such that λ is p-restricted.

Now we show that in steps (2) and (3) we construct the set A as the set of dominant weights λ with $m(\lambda) \leqslant M$.

First note that for $\omega \leqslant \lambda$ dominant we have $m(\omega) \leqslant m(\lambda)$.

Furthermore let $\lambda = a_1\omega_1 + \cdots + a_l\omega_l$ be a dominant weight which is not 2-restricted, i.e., one $a_i \geqslant 2$. Then $\lambda - \alpha_i$ is also a dominant weight. This follows from the fact that the coefficients of the simple root α_i expressed as linear combination of the fundamental weights is given by the i-th column of the Cartan matrix (which has entries 2 on the diagonal and non-positive entries elsewhere).

So, for a weight λ with one of the coefficients $a_i \geqslant 2k$, $k \in \mathbb{N}$, there are at least k smaller dominant weights and so $m(\lambda) > k$. This shows that we are constructing a finite set and that we will find the interval I in step (3).

Now we show by induction on the value n of γ, starting from the lower bound $n = a$ of the interval I, that for a dominant weight λ with $\gamma(\lambda) = n$ we have $m(\lambda) > M$.

This is clear for $n \in I$ by construction of the interval I in step (3).

So, assume that $\gamma(\lambda) > a + \max(\gamma(\alpha_i) \mid 1 \leqslant i \leqslant l)$. If λ is 2-restricted we have found in step (2) that $m(\lambda) > M$ (it is not in A by construction of I). Otherwise one coefficient of lambda, say a_i, is $\geqslant 2$. Then $\lambda - \alpha_i$ is dominant and we have $a \leqslant \gamma(\lambda - \alpha_i) < \gamma(\lambda)$. By induction hypothesis we get $M < m(\lambda - \alpha_i) < m(\lambda)$. □

In Table 1 we fix some bound M for each irreducible root system of rank at most 11, respectively 17 in case A_l.

For each irreducible root system listed in Table 1 we used the given M as input for Algorithm 4.3. For each weight λ in the output set A and for all primes p for which λ

Table 1: Chosen degree bounds for groups of small rank

Type	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
M	400	500	1000	2500	2800	3000	4000	6000	10000	12000
Type	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}				
M	3000	3000	3000	3000	3000	3000				
Type	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}
M	300	700	1000	2000	4000	5000	7000	8000	10000	12000
Type		C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
M		1000	2000	2500	4000	6000	10000	10000	10000	12000
Type			D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}
M			2000	3000	4000	5000	10000	15000	18000	20000
Type	G_2		F_4		E_6	E_7	E_8			
M	500		12000		50000	100000	100000			

is p-restricted we computed the exact weight multiplicities of $L(\lambda)$ using the techniques described in Section 3.

We remark that the sets A from Algorithm 4.3 are much larger than the list of weights which actually lead to representations of dimension $\leqslant M$, since we used a very rough lower bound for these dimensions. But for weights of representations of much larger degree we usually find quite fast some weight multiplicities > 1 which gives better lower bounds for the dimension. We stop the computation of weight multiplicities when this better estimate becomes larger than M. In this way almost all of the computation time for our result was actually spent on those weights which appear in our result tables.

The bounds M were chosen such that the results presented below could be computed interactively with our programs within about two days using 10 computers in parallel. The types A_l, $12 \leqslant l \leqslant 17$, were added upon request of Gunter Malle, who asked to cover in this note all representations of degree $\leqslant 2\dim(G)$ for a specific application.

Here is our main result.

Theorem 4.4. *For any type of root system and number M as given in Table 1 and all primes p the Tables 6.6 to 6.53 list the p-restricted weights λ such that the representation $L(\lambda)$ of the algebraic group G over $\bar{\mathbb{F}}_p$ has degree at most M. The exact degree of $L(\lambda)$ is also given. Furthermore we describe the centers of the groups G and the action of the fundamental weights on the center in 6.2. This allows to determine the kernels of the representations $L(\lambda)$. The Frobenius-Schur indicators in case $p \neq 2$ are given by 6.3.*

As mentioned above we have actually computed the exact weight multiplicities for the representations appearing in the tables of Section 6. It would take too much space to print this in detail but the results are available upon request from the author.

Our table for type F_4 includes a description of the representations of the Ree group ${}^2F_4(2)$. The related finite simple group is the commutator subgroup which is of index 2. The restrictions of the irreducible representations of ${}^2F_4(2)$ in characteristic 2 remain irreducible, except that with highest weight $\lambda = (1100)$ which splits into two representations of degree 2048, see the Modular Atlas [11].

The types considered above do not include A_1, i.e., $G = SL_2(\bar{\mathbb{F}}_p)$. The reason is that this case is easy to describe systematically. This seems to be well known but also follows immediately from 4.1 since all weight multiplicities are at most 1 in this case, see [9, 7.2].

Remark 4.5. If G is of type A_1 then the representation $L(k\omega_1)$ with $0 \leqslant k \leqslant p-1$ has degree $k+1$. For odd p the center of G is non-trivial and of order 2. It is contained in the kernel of $L(k\omega_1)$ if and only if k is even.

5. *Representations of small degree for groups of large rank*

In this section we consider classical groups of large rank. We prove the following result.

Theorem 5.1. *Let G be of classical type and l be the rank of G. Set $M = l^3/8$ if G is of type A_l and $M = l^3$ otherwise. If $l > 11$ then all p-restricted weights λ such that the highest weight representation $L(\lambda)$ of G has dimension at most M are given in Table 2. The table also includes the dimensions of these modules. (Note that the case of type B_l and $p = 2$ is included in case C_l and $p = 2$.)*

The fundamental weights are labeled as explained in 6.1.

Table 2: Small degrees for classical groups of large rank

type	λ	p	degree
A_l	0	all	1
	ω_1, ω_l	all	$l+1$
	ω_2, ω_{l-1}	all	$l(l+1)/2$
	$2\omega_1$, $2\omega_l$	all	$(l+1)(l+2)/2$
	$\omega_1+\omega_l$	$p \mid l+1$	l^2+2l-1
	$\omega_1+\omega_l$	$p \nmid l+1$	l^2+2l
B_l	0	all	1
	ω_l	$\neq 2$	$2l+1$
	ω_{l-1}	$\neq 2$	$2l^2+l$
	$2\omega_l$	$p \mid 2l+1$	$2l^2+3l-1$
	$2\omega_l$	$p \nmid 2l+1$	$2l^2+3l$
C_l	0	all	1
	ω_l	all	$2l$
	ω_{l-1}	$p \mid l$	$2l^2-l-2$
	ω_{l-1}	$p \nmid l$	$2l^2-l-1$
	$2\omega_l$	all	$2l^2+l$
D_l	0	all	1
	ω_l	all	$2l$
	ω_{l-1}	2	$2l^2-l-\gcd(2,l)$
	ω_{l-1}	$\neq 2$	$2l^2-l$
	$2\omega_l$	$p \mid l$	$2l^2+l-2$
	$2\omega_l$	$p \nmid l$	$2l^2+l-1$

Note that the corresponding result for G of rank $l \leqslant 11$ is included in Theorem 4.4.

Proof. We first prove that the weights which don't appear in our table correspond to representations of degree larger than M.

(Case B_l, C_l) Let $\lambda = a_1\omega_1 + \cdots a_l\omega_l$ be a dominant weight. If some $a_i \neq 0$ then the W-stabilizer of λ is contained in a reflection subgroup of type $B_{i-1} \times A_{l-i}$, see 4.2. Hence, the W-orbit of λ and so $\dim(L(\lambda))$ is at least

$$b_i := (2^l l!)/(2^{i-1}(i-1)! \cdot (l-i+1)!) = 2^{l-i+1} \begin{pmatrix} l \\ i-1 \end{pmatrix}.$$

For $l \geqslant 12$ and $1 \leqslant i \leqslant l-2$ we have $b_i > l^3$. If $a_{l-1} \neq 0$ and $a_l \neq 0$ then the stabilizer of λ is contained in a reflection group of type B_{l-2}, whose index is $> l^3$ for all $l \geqslant 6$.

This shows that only weights with at most one non-zero coefficient, either a_{l-1} or a_l, can appear in our list. If $a_{l-1} \geqslant 2$ then, as explained in the proof of 4.3, $\lambda - \alpha_{l-1} \leqslant \lambda$ is also a dominant weight. But this has coefficient 1 at ω_{l-2}. Using the estimate as above for this smaller weight and Premet's theorem 4.1 (note that the λ considered now is not 2-restricted) we see again that $\dim(L(\lambda)) > l^3$. A similar argument shows that $a_l < 3$ for weights in our list.

So, the only weights which could (and actually do) lead to degrees $\leqslant l^3$ are 0, ω_l, ω_{l-1} and $2\omega_l$.

(Case D_l) Here we find the relevant λ with very similar arguments as in case B_l and C_l.

(Case A_l) Because of the symmetry of the Dynkin diagram the weights $a_1\omega_1 + \cdots + a_l\omega_l$ and $a_l\omega_1 + a_{l-1}\omega_2 + \cdots + a_1\omega_l$ must describe representations of equal degree (in fact they are dual to each other). We can again use very similar arguments as above to find the relevant weights for our list. (Here we rule out $3\omega_1$ and $\omega_1 + \omega_2$ by observing $3\omega_1 - 2\alpha_1 - \alpha_2 = \omega_3 = \omega_1 + \omega_2 - \alpha_1 - \alpha_2$.)

It remains to determine the exact degrees of the $L(\lambda)$ in our table. Probably they are known to the experts but we could not find references for all cases. Type A_l and $L(\omega_l)$ and $L(\omega_{l-1})$ for the other types are contained in [14, 5.4.11] and [9, 25.5, Ex. 8].

We include a proof for the degrees of $L(2\omega_l)$ and $L(\omega_{l-1})$ in types B_l, C_l and D_l following hints of K. Magaard and G. Hiß.

The idea is to use explicit modules. For all types we know the G-modules $V(X_l) := L(\omega_l)$, $X \in \{A,B,C,D\}$. These are the natural modules of SL_{l+1}, SO_{2l+1}, Sp_{2l} or SO_{2l}, respectively. We determine the constituents of $V(X_l) \otimes V(X_l)$ for $X \in \{B,C,D\}$. Note that for these types $V(X_l)$ is self-dual. Since the case $p = 2$ is covered by the references above ($2\omega_l$ is not 2-restricted), we assume in the rest of the proof that p is odd.

We will use the following general remarks. If V is an indecomposable $\bar{\mathbb{F}}_p G$-module then $V \otimes V^*$ has the trivial module as direct summand if and only if $p \nmid \dim(V)$ (here V^* denotes the dual module). In that case there is exactly one trivial direct summand, see [1, 3.1.9] for a proof. If $v_1, \ldots, v_n$ is a basis of V then $V \otimes V$ has two submodules $S^2(V)$ with basis $\{v_i \otimes v_j + v_j \otimes v_i \mid 1 \leqslant i < j \leqslant n\} \cup \{v_i \otimes v_i \mid 1 \leqslant i \leqslant n\}$ and $\Lambda^2(V)$ with basis $\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \leqslant i < j \leqslant n\}$. Since $p \neq 2$ we have $V \otimes V = S^2(V) \oplus \Lambda^2(V)$.

(Case C_l) Let $\{v_i, v_i' \mid 1 \leqslant i \leqslant l\}$ be a basis of the natural symplectic module $V(C_l)$ such that for the symplectic form $(\cdot,\cdot)$ we have $(v_i, v_i') = 1 = -(v_i', v_i)$, $1 \leqslant i \leqslant l$ and $(v_i, v_j) = 0 = (v_i, v_j')$ for all $i \neq j$. Then the vector $\sum_{i=1}^{l}(v_i \otimes v_i' - v_i' \otimes v_i) \in \Lambda^2(V)$ is invariant under the symplectic group. If $p \nmid 2l$ this vector must span the unique trivial direct summand mentioned above.

The group G contains a subgroup isomorphic to $SL_l(\bar{\mathbb{F}}_p)$. An element acts on the subspace of $V(C_l)$ spanned by $v_1, \ldots, v_l$ and by its inverse on the subspace spanned by $v_1', \ldots v_l'$. Hence $V(C_l)$ restricted to this subgroup is isomorphic to $V(A_{l-1}) \oplus V(A_{l-1})^*$. Restricting the representation $S^2(V(C_l))$ to this subgroup we find the decomposition $S^2(V(A_{l-1})) \oplus$

$S^2(V(A_{l-1})^*) \oplus (V(A_{l-1}) \otimes V(A_{l-1})^*)$, see [5, I,12.]. Using the known degrees for case A_{l-1}, we see that this contains at most 2 trivial constituents. The similar restriction to a subgroup of type C_{l-1} shows by induction that one irreducible constituent of $S^2(V(C_l))$ has degree at least $2(l-1)^2+l-1$. Comparing degrees we see that the three non-trivial constituents in the restriction to SL_l must lie in a single constituent of $S^2(V(C_l))$. To summarize, we have at most one non-trivial and two trivial constituents. If there are trivial constituents then one must be in the socle, i.e., there must be an invariant vector. This can only be the one we see in the $(V(A_{l-1}) \otimes V(A_{l-1})^*)$-summand of the restriction to SL_l. We can write down such a vector and check that it is not invariant under the whole group G; apply an element which does not leave the space spanned by $v_1, \ldots, v_l$ invariant. We have proved that $S^2(V(C_l))$ is irreducible.

We can argue very similarly for $\Lambda^2(V(C_l))$. If $p \nmid l$ we find that it is a direct sum of an irreducible and the trivial module found above. If $p \mid l$ we find that there is one non-trivial constituent, exactly one trivial constituent in the socle and at most two trivial constituents. Since in this case the trivial constituent in the socle is not a direct summand there must be a second trivial constituent in the head, by duality.

If $V(C_l)$ has a highest weight vector v then $v \otimes v$ is contained in $S^2(V(C_l))$ and has weight $2\omega_l$. Hence $S^2(V(C_l)) \cong L(2\omega_l)$. The other constituents of the tensor product must correspond to dominant weights smaller than $2\omega_l$, there are only two of them, $2\omega_l - \alpha_l = \omega_{l-1}$ and 0. We get that the non-trivial constituent of $\Lambda^2(V(C_l))$ is isomorphic to $L(\omega_{l-1})$.

(Case D_l) This can be handled by almost exactly the same arguments as the case C_l. Here the trivial submodule is contained in $S^2(V(D_l))$.

(Case B_l) In this case we consider the restrictions of $S^2(V(B_l))$ to subgroups H_r of type $B_r + D_{l-r}$, $0 \leqslant r < l$. This leads to decompositions $S^2(V(B_r)) \oplus S^2(V(D_{l-r})) \oplus (V(D_{l-r}) \otimes V(B_r))$. Comparing this decomposition for $r = 0$ and $r = l/2$ or $r = (l-1)/2$, respectively, and using the results for type D_{l-r} and induction we see as in type C_l that $S^2(V(B_l))$ has only one non-trivial and maybe a few trivial constituents. There is an r such that $p \nmid (l-r)$ and $p \nmid (2r+1)$. The decomposition above for this r shows that there are at most two trivial constituents. Furthermore, as in type D_l we find a trivial submodule. Either this is a direct summand (if $p \nmid (2l+1)$) then it is the only trivial constituent or otherwise there is a second constituent in the head of $S^2(V(B_l))$. The argument for $\Lambda^2(V(B_l))$ is again very similar.

This finishes the proof. ☐

It would be interesting if the degrees in our list could be determined more systematically in the framework of highest weight modules. Then one could work out systematically generalizations of Theorem 5.1 where the bound M is substituted by any fixed polynomial in l.

The following Corollary completes the list in [8] which gives all representations of $G(q)$ in non-defining characteristic of degree at most 250.

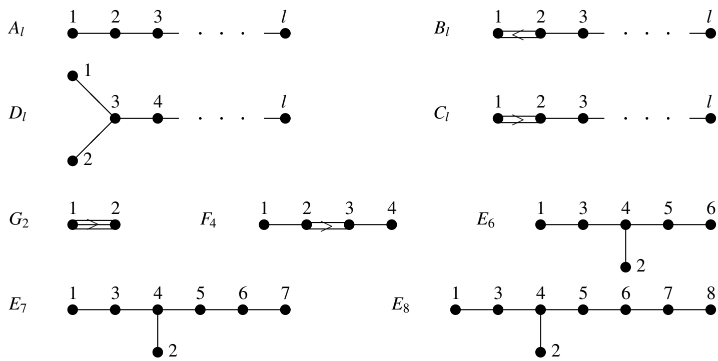
Corollary 5.2. *For any simple G over $\bar{\mathbb{F}}_p$ we find all p-restricted weights λ of G such that $L(\lambda)$ has degree $\leqslant 250$ in the tables given in Theorems 4.4 and 5.1. Taking Theorem 2.2 into account this determines all $L(\lambda)$ with degree $\leqslant 250$.*

6. *Appendix: Tables for groups of small rank*

In this section we give the detailed lists for Theorem 4.4. We start by introducing some notation and describe the centers of the groups G and the action of the fundamental weights on the center.

6.1. *Ordering of fundamental weights*

For the irreducible types of root systems we choose the following ordering for the simple roots α_i, the corresponding coroots $\alpha_i^\vee$ and the fundamental weights ω_i, $1 \leqslant i \leqslant l$. We show the Dynkin diagrams with the node of α_i labeled by i. (This is the labeling we use in all databases of the CHEVIE [6] project.)



6.2. *Action of fundamental weights on the center of G*

We give for each irreducible type of root system the values of $\omega_i(z)$ for z in the center $Z(G)$ of G.

To compute this we use that $Z(G)$ is contained in a maximal torus T of G. Such a T is isomorphic to $(\sum_{i=1}^{l} \mathbb{Z}\alpha_i^\vee) \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p^\times$ and $Z(G)$ consists of those $z \in T$ with $\alpha_i(z) = 1$ for all $1 \leqslant i \leqslant l$. So, we have to solve a system of equations given by the Cartan matrix of the root system.

We denote $\zeta_m \in \bar{\mathbb{F}}_p^\times$ an element whose multiplicative order is m. For $n \in \mathbb{N}$ we write $n_{p'}$ for the largest divisor of n which is prime to p. For elements $z \in Z(G)$ we write $\underline{\omega}(z) := (\omega_1(z), \ldots, \omega_l(z))$.

(A_l) $Z(G)$ is cyclic of order $m = (l+1)_{p'}$. It contains a generator z such that $\underline{\omega}(z) = (\zeta_m, \zeta_m^2, \ldots \zeta_m^l)$.

(B_l) $Z(G)$ is cyclic of order $m = \gcd(2, p+1)$. For the generator z we have $\underline{\omega}(z) = (\zeta_m, 1, \ldots, 1)$.

(C_l) $Z(G)$ is cyclic of order $m = \gcd(2, p+1)$. For the generator z we have $\underline{\omega}(z) = (\zeta_m^l, \zeta_m^{l-1}, \ldots, 1, \zeta_m)$

(D_l, l odd) $Z(G)$ is cyclic of order $m = 4$ for odd p and $m = 1$ for $p = 2$. There is a generator z such that $\underline{\omega}(z) = (\zeta_m, \zeta_m^3, \zeta_m^2, 1, \zeta_m^2, 1, \zeta_m^2, \ldots, 1, \zeta_m^2)$.

(D_l, l even) $Z(G)$ is elementary abelian of order m^2 with $m = 2$ if p is odd and $m = 1$ if $p = 2$. There are generators z_1 and z_2 such that $\underline{\omega}(z_1) = (\zeta_m, 1, 1, \zeta_m, 1, \zeta_m, \ldots, 1, \zeta_m)$ and $\underline{\omega}(z_2) = (1, \zeta_m, 1, \zeta_m, 1, \zeta_m, \ldots, 1, \zeta_m)$. Here $z = z_1 z_2$ is the element such that $G/\langle z \rangle \cong SO_{2l}(\bar{\mathbb{F}}_p)$.

(E_6) $Z(G)$ is cyclic of order $m = 3$ if $p \neq 3$ and $m = 1$ if $p = 3$. There is a generator z such that $\underline{\omega}(z) = (\zeta_m, 1, \zeta_m^2, 1, \zeta_m, \zeta_m^2)$.

(E_7) $Z(G)$ is cyclic of order $m = \gcd(2, p+1)$. For the generator z we have $\underline{\omega}(z) = (1, \zeta_m, 1, 1, \zeta_m, 1, \zeta_m)$.

(G_2, F_4, E_8) $Z(G)$ is trivial.

6.3. *Frobenius-Schur indicators*

If p is odd we can determine the Frobenius-Schur indicators of the representations in our lists using a result of Steinberg, see [20, Lemmas 78 and 79].

If G is of type A_l or of type D_l with odd l or of type E_6 then the representations $L(\sum_{i=1}^{l} a_i\omega_i)$ and $L(\sum_{i=1}^{l} a_{\tau(i)}\omega_i)$, where the permutation τ is given by the automorphism of order two of the Dynkin diagram, are dual to each other. For other G all $L(\lambda)$ are self-dual.

If $L(\lambda)$ is self-dual (and recall that p is odd) then its Frobenius-Schur indicator can be computed by evaluating $\lambda(z)$ for a certain central element z of G.

The element z is described by Steinberg in the form $\prod_\alpha h_\alpha(-1)$ where the product is over all positive roots. This is the (central) torus element z on which each weight μ evaluates as $(-1)^{\langle\mu,2\rho^\vee\rangle}$ where $2\rho^\vee$ is the sum of all positve coroots and $\langle\cdot,\cdot\rangle$ is the pairing between X and Y. So, the element z in the notation of Appendix 6.2 has i-th coordinate -1 if the i-th coefficient of $2\rho^\vee$ in the basis of simple coroots is odd and $+1$ otherwise. Using the description of $2\rho^\vee$ in [2, Planche I to IX] we can identify z with one of the center elements described above.

In types A_l with even l, B_l with $l \equiv 0,3 \pmod 4$, D_l with $l \equiv 0,1 \pmod 4$, G_2, F_4, E_6 and E_8 we get $z = 1$. In types D_l with $l \equiv 2 \pmod 4$ we get $z = z_1z_2 = (-1,-1,1,1,\ldots,1)$ and in the other cases z is the only element of order 2 in the center (see 6.2).

6.4. *An example*

As an example how to read the data in this Appendix, let us determine all irreducible representations in defining characteristic for groups of type $B_3(q) \cong \mathrm{Spin}_7(q)$, $q = p^f$ which have degree 448:

We apply Theorems 2.2 and 2.3.

We first need to compute all factorizations of 448 into factors > 1 which appear as degrees in 6.23. Although many divisors of 448 appear as degree the only such factorizations are $448 = 7\cdot 64 = 7\cdot 8\cdot 8$.

Now we have a closer look at 6.23.

If $p = 2$ then there is no irreducible representation of degree 7. And the listed representations of degree 448 do not correspond to 2-restricted weights. So, in characteristic 2 there are no irreducible representations of this degree.

For $p \neq 2$ we have representations with p-restricted weights of dimension 7 and 8. If $q \geqslant p^3$ there are irreducible representations $L_{i,j,k}$ of degree 448 with highest weight of form $p^i\omega_3 + p^j\omega_1 + p^k\omega_1$, where $0 \leqslant i,j,k \leqslant f-1$ and i,j,k are pairwise different.

For $p = 3$ there is no irreducible representation of degree 64 and none with p-restricted weight of degree 448. So, there are no further irreducible representations of degree 448 in this case.

If $p = 5$ there is $L(\omega_3)$ of degree 7 and $L(\omega_1+\omega_2)$ of degree 64. If $q = 5^f$ we find for any $0 \leqslant j,k \leqslant f-1$, $j \neq k$, the irreducible representation $L'_{j,k} := L(5^j(\omega_1+\omega_2) + 5^k\omega_3)$ restricted to $B_3(q)$ of degree 448. Furthermore we see in 6.23 that $L(\omega_1+3\omega_3)$ and $L(2(\omega_1+\omega_3))$ have degree 448.

For larger p there is no representation of degree 64 in our list. We only find $L(\omega_1+3\omega_3)$ if $p \neq 11$ and in case $p = 7$ also $L(3\omega_1+\omega_2)$.

For all representations found above the prime p is odd and so the center of G and of $G(q)$ is of order 2. Using 6.2 we see that for the non-trivial element z in the center we have $\omega_1(z) = -1$ and $\omega_i(z) = 1$ for $i = 2,3$. Applying this to the weights listed above we find that exactly $L(2(\omega_1+\omega_3))$ in case $p = 5$ and $L_{i,j,k}$ for $p \neq 2$ are not faithful. From 6.3 we see that the Frobenius-Schur indicator is $+1$ in all cases.

6.5. *Reading the tables*

In the tables below we denote a weight $a_1\omega_1 + \cdots + a_l\omega_l$ by $(a_1, \ldots, a_l)$. When all a_i can be written with a single digit we also suppress the commas.

In type A_l the representations $L(a_1\omega_1 + \cdots + a_l\omega_l)$ and $L(a_l\omega_1 + a_{l-1}\omega_2 + \cdots + a_1\omega_l)$ are dual to each other, in particular they have the same dimension. In 6.6 to 6.21 we save some space by only including one of such a pair of representations.

6.6. *Case* A_2, $M = 400$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(00)	all	120	(35)	$\neq 7$	255	(67)	13
3	(01)	all	123	(48)	13	260	(47)	$\neq 11$
6	(02)	all	125	(44)	$\neq 7$	262	(6,12)	19
7	(11)	3	126	(57)	13	267	(48)	11
8	(11)	$\neq 3$	127	(66)	13	267	(7,11)	19
10	(03)	all	132	(27)	all	270	(2,11)	$\neq 13$
15	(04)	all	136	(0,15)	all	270	(8,10)	19
15	(12)	all	143	(1,10)	all	271	(99)	19
18	(13)	5	153	(0,16)	all	273	(56)	$\neq 11$
19	(22)	5	154	(36)	all	276	(0,22)	all
21	(05)	all	159	(29)	11	276	(57)	11
24	(13)	$\neq 5$	162	(28)	$\neq 11$	279	(66)	11
27	(22)	$\neq 5$	162	(45)	7	280	(39)	$\neq 11,13$
28	(06)	all	165	(45)	$\neq 7$	288	(1,15)	$\neq 17$
33	(15)	7	168	(1,11)	$\neq 13$	297	(1,21)	23
35	(14)	all	168	(1,15)	17	300	(0,23)	all
36	(07)	all	171	(0,17)	all	312	(2,12)	all
36	(24)	7	171	(38)	11	315	(48)	$\neq 11,13$
37	(33)	7	179	(47)	11	316	(2,20)	23
39	(23)	5	181	(2,14)	17	323	(1,16)	all
42	(23)	$\neq 5$	183	(56)	11	325	(0,24)	all
45	(08)	all	190	(0,18)	all	330	(3,10)	$\neq 13$
48	(15)	$\neq 7$	192	(37)	$\neq 11$	333	(3,19)	23
55	(09)	all	192	(3,13)	17	336	(57)	$\neq 11,13$
60	(24)	$\neq 7$	195	(1,12)	all	339	(3,11)	13
63	(16)	all	195	(29)	$\neq 11$	343	(66)	$\neq 11,13$
63	(33)	5	201	(4,12)	17	348	(4,18)	23
64	(33)	$\neq 5,7$	207	(1,17)	19	351	(0,25)	all
66	(0,10)	all	208	(5,11)	17	351	(2,15)	17
71	(25)	7	210	(0,19)	all	354	(49)	11
75	(19)	11	210	(46)	$\neq 11$	357	(2,13)	all
75	(34)	7	213	(6,10)	17	360	(1,17)	$\neq 19$
78	(0,11)	all	215	(2,11)	13	360	(4,10)	13
80	(17)	all	215	(55)	7	361	(5,17)	23
81	(25)	$\neq 7$	216	(55)	$\neq 7,11$	370	(58)	11
82	(28)	11	216	(79)	17	372	(6,16)	23
87	(37)	11	217	(88)	17	375	(3,14)	17
90	(34)	$\neq 7$	222	(2,16)	19	375	(49)	$\neq 11,13$
90	(46)	11	224	(1,13)	all	375	(59)	13
91	(0,12)	all	231	(0,20)	all	378	(0,26)	all
91	(55)	11	231	(2,10)	$\neq 13$	378	(67)	11
99	(18)	all	231	(3,10)	13	381	(7,15)	23
102	(1,11)	13	234	(38)	$\neq 11$	384	(3,11)	$\neq 13$
105	(0,13)	all	235	(3,15)	19	384	(68)	13
105	(26)	all	243	(49)	13	387	(77)	13
111	(2,10)	13	246	(4,14)	19	388	(8,14)	23
114	(35)	7	251	(58)	13	393	(9,13)	23
117	(44)	7	252	(39)	11	395	(4,13)	17
118	(39)	13	253	(0,21)	all	396	(10,12)	23
120	(0,14)	all	255	(1,14)	all	397	(11,11)	23
120	(19)	$\neq 11$	255	(5,13)	19	399	(1,18)	all

6.7. *Case* A_3, $M = 500$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(000)	all	120	(007)	all	285	(303)	7
4	(001)	all	120	(104)	≠7	286	(0,0,10)	all
6	(010)	all	124	(203)	7	294	(212)	3
10	(002)	all	126	(022)	≠5	299	(222)	5
14	(101)	2	140	(014)	all	300	(122)	7
15	(101)	≠2	140	(031)	≠5	300	(212)	≠3,7
16	(011)	3	140	(112)	≠3	300	(303)	≠7
19	(020)	3	149	(024)	7	315	(016)	all
20	(003)	all	156	(121)	3	334	(131)	7
20	(011)	≠3	160	(203)	≠7	336	(060)	≠7
20	(020)	≠3	165	(008)	all	360	(019)	11
32	(102)	5	173	(113)	5	360	(024)	≠7
35	(004)	all	175	(121)	≠3	360	(122)	≠5,7
36	(102)	≠5	180	(033)	7	364	(0,0,11)	all
44	(111)	3	184	(131)	5	380	(115)	7
45	(012)	all	189	(105)	all	380	(213)	5
50	(030)	all	192	(023)	5	384	(131)	≠5,7
52	(013)	5	196	(050)	all	396	(107)	all
56	(005)	all	206	(042)	7	416	(025)	7
58	(111)	5	211	(113)	7	420	(108)	11
60	(021)	all	216	(015)	≠7	420	(114)	all
64	(111)	≠3,5	220	(009)	all	420	(205)	all
68	(022)	5	220	(122)	5	420	(313)	5
69	(202)	3	224	(023)	≠5	439	(028)	11
70	(103)	all	224	(051)	7	440	(017)	all
80	(031)	5	231	(060)	7	455	(0,0,12)	all
83	(202)	5	235	(032)	5	460	(033)	5
84	(006)	all	236	(212)	7	476	(132)	5
84	(013)	≠5	256	(113)	≠5,7	480	(033)	≠5,7
84	(202)	≠3,5	260	(041)	5	480	(124)	7
85	(040)	5	260	(204)	7	484	(151)	7
100	(104)	7	270	(204)	≠7	496	(304)	7
105	(040)	≠5	280	(032)	≠5	500	(304)	≠7
116	(015)	7	280	(041)	≠5			
116	(112)	3	280	(106)	all			

6.8. *Case* A_4, $M = 1000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000)	all	185	(0022)	5	510	(0112)	3
5	(0001)	all	195	(0201)	3	535	(1013)	5
10	(0010)	all	199	(2002)	3	540	(0104)	$\neq 7$
15	(0002)	all	200	(2002)	$\neq 3,7$	545	(0202)	5
23	(1001)	5	210	(0006)	all	560	(0031)	$\neq 5$
24	(1001)	$\neq 5$	210	(0201)	$\neq 3,5$	560	(0202)	$\neq 3,5$
30	(0011)	3	224	(0013)	$\neq 5$	560	(1005)	all
35	(0003)	all	235	(0111)	5	615	(1021)	5
40	(0011)	$\neq 3$	255	(0031)	5	640	(0033)	7
40	(0101)	2	280	(0103)	all	670	(0023)	5
45	(0020)	3	280	(0111)	$\neq 3,5$	670	(1021)	7
45	(0101)	$\neq 2$	305	(0120)	5	683	(1022)	5
50	(0020)	$\neq 3$	315	(0120)	$\neq 5$	700	(0112)	$\neq 3$
51	(0110)	3	315	(1004)	all	700	(0301)	all
65	(1002)	3	320	(0040)	5	715	(0009)	all
70	(0004)	all	325	(0015)	7	720	(0015)	$\neq 7$
70	(1002)	$\neq 3$	330	(0007)	all	720	(1021)	$\neq 5,7$
74	(0110)	2	365	(0130)	5	765	(0203)	7
75	(0110)	$\neq 2,3$	375	(1102)	3	794	(1111)	5
103	(0102)	5	381	(0220)	5	826	(0042)	7
105	(0012)	all	390	(0202)	3	835	(0211)	3
121	(0013)	5	395	(0104)	7	840	(0023)	$\neq 5$
126	(0005)	all	410	(1012)	7	855	(0121)	3
126	(0102)	$\neq 5$	410	(1102)	5	875	(2004)	all
135	(1011)	3	420	(0014)	all	895	(0113)	5
145	(1003)	7	420	(0022)	$\neq 5$	924	(1006)	all
160	(1003)	$\neq 7$	435	(1102)	7	945	(0105)	all
160	(1011)	2	445	(2003)	7	945	(1013)	$\neq 5$
165	(0111)	3	450	(1012)	$\neq 7$	949	(1111)	7
170	(0201)	5	450	(2003)	$\neq 7$	955	(0032)	5
174	(0021)	3	470	(0024)	7	980	(0130)	$\neq 5$
175	(0021)	$\neq 3$	476	(1111)	3	999	(3003)	7
175	(0030)	all	480	(1102)	$\neq 3,5,7$	1000	(3003)	$\neq 7$
175	(1011)	$\neq 2,3$	490	(0040)	$\neq 5$			
176	(2002)	7	495	(0008)	all			

6.9. *Case A_5, $M = 2500$*

(Recall the remark in 6.5.)

deg	λ	p
1	(00000)	all
6	(00001)	all
15	(00010)	all
20	(00100)	all
21	(00002)	all
34	(10001)	2
34	(10001)	3
35	(10001)	$\neq$2,3
50	(00011)	3
56	(00003)	all
70	(00011)	$\neq$3
78	(01001)	5
84	(01001)	$\neq$5
90	(00020)	3
90	(00101)	2
105	(00020)	$\neq$3
105	(00101)	$\neq$2
114	(10002)	7
120	(10002)	$\neq$7
126	(00004)	all
126	(00110)	3
141	(00200)	3
154	(01010)	2
175	(00200)	$\neq$3
188	(01010)	5
189	(01010)	$\neq$2,5
204	(00110)	2
210	(00012)	all
210	(00110)	$\neq$2,3
246	(00013)	5
246	(01002)	3
252	(00005)	all
258	(00102)	5
279	(10011)	3
280	(01002)	$\neq$3
315	(10003)	all
336	(00102)	$\neq$5
363	(10011)	5
369	(10011)	7
384	(10011)	$\neq$3,5,7
400	(10101)	2
404	(20002)	7
405	(20002)	$\neq$7
414	(00021)	3
420	(00021)	$\neq$3

deg	λ	p
426	(00022)	5
440	(02001)	3
462	(00006)	all
474	(00111)	3
490	(00030)	all
504	(00013)	$\neq$5
520	(10101)	7
540	(10101)	$\neq$2,7
560	(02001)	$\neq$3
606	(01003)	7
630	(00201)	5
630	(01011)	3
666	(00031)	5
666	(01101)	3
700	(10004)	all
708	(00111)	5
720	(00201)	3
720	(01003)	$\neq$7
720	(01011)	2
786	(00015)	7
792	(00007)	all
804	(11002)	3
813	(01020)	5
840	(00103)	all
840	(00201)	$\neq$3,5
840	(01011)	$\neq$2,3
896	(00111)	$\neq$3,5
924	(01101)	2
951	(00040)	5
960	(00300)	5
960	(01110)	3
980	(00300)	$\neq$5
1050	(00014)	all
1050	(01101)	$\neq$2,3
1050	(20003)	all
1071	(01020)	3
1078	(10012)	5
1098	(00120)	5
1134	(00022)	$\neq$5
1134	(10012)	$\neq$5
1170	(00120)	3
1176	(00120)	$\neq$3,5
1176	(01020)	$\neq$3,5
1194	(00104)	7
1224	(10013)	5

deg	λ	p
1246	(00130)	5
1251	(00024)	7
1260	(11002)	$\neq$3
1287	(00008)	all
1338	(10102)	5
1365	(00210)	5
1386	(10005)	all
1420	(01110)	5
1431	(02002)	7
1449	(00210)	3
1470	(00210)	$\neq$3,5
1506	(00220)	5
1539	(00202)	3
1569	(02002)	3
1575	(01004)	all
1674	(00112)	3
1686	(00310)	5
1701	(10102)	$\neq$5
1751	(00400)	5
1764	(00031)	$\neq$5
1764	(00040)	$\neq$5
1800	(00104)	$\neq$7
1800	(02002)	$\neq$3,7
1876	(00033)	7
1890	(00023)	5
1960	(01110)	$\neq$3,5
1974	(10021)	3
1980	(00015)	$\neq$7
1995	(20004)	5
2002	(00009)	all
2024	(11011)	3
2061	(01012)	7
2100	(03001)	7
2106	(01102)	3
2205	(10021)	$\neq$3
2268	(10006)	11
2290	(30003)	5
2310	(10022)	5
2310	(20004)	$\neq$5
2364	(10201)	5
2415	(00202)	5
2430	(01012)	$\neq$7
2430	(10111)	3

6.10. *Case* A_6, $M = 2800$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000)	all	490	(000110)	$\neq$2,3	1428	(010011)	3
7	(000001)	all	490	(000200)	$\neq$3	1520	(000031)	5
21	(000010)	all	493	(010002)	7	1575	(010003)	all
28	(000002)	all	540	(010002)	$\neq$7	1701	(011001)	3
35	(000100)	all	553	(000102)	5	1709	(000015)	7
47	(100001)	7	560	(100003)	all	1716	(000007)	all
48	(100001)	$\neq$7	581	(001010)	5	1771	(000111)	5
77	(000011)	3	588	(001010)	$\neq$2,5	1827	(010011)	5
84	(000003)	all	687	(200002)	3	1863	(000201)	5
112	(000011)	$\neq$3	707	(001002)	3	1876	(010011)	7
133	(010001)	2	707	(100011)	2	1907	(001003)	7
133	(010001)	3	735	(100011)	$\neq$2,3	1953	(110002)	3
140	(010001)	$\neq$2,3	735	(200002)	$\neq$3	1967	(002001)	3
161	(000020)	3	736	(001100)	2	1967	(200003)	5
175	(000101)	2	756	(000102)	$\neq$5	2016	(010011)	$\neq$3,5,7
189	(100002)	all	783	(001100)	5	2100	(000103)	all
196	(000020)	$\neq$3	784	(001100)	$\neq$2,3,5	2106	(000201)	3
203	(001001)	5	840	(001002)	$\neq$3	2155	(001011)	3
210	(000004)	all	861	(000021)	3	2156	(200003)	$\neq$5
210	(000101)	$\neq$2	875	(000022)	5	2198	(010101)	2
224	(001001)	$\neq$5	882	(000021)	$\neq$3	2310	(000014)	all
266	(000110)	3	924	(000006)	all	2331	(100012)	3
344	(010010)	5	1008	(000013)	$\neq$5	2352	(000111)	$\neq$3,5
357	(000200)	3	1050	(020001)	3	2387	(010020)	3
378	(000012)	all	1113	(100101)	2	2394	(001101)	3
391	(010010)	3	1148	(000111)	3	2400	(001003)	$\neq$7
392	(010010)	$\neq$3,5	1148	(020001)	7	2400	(001011)	2
393	(001100)	3	1176	(000030)	all	2415	(000040)	5
448	(001010)	2	1211	(100101)	5	2450	(100012)	$\neq$3
455	(000013)	5	1260	(020001)	$\neq$3,7	2611	(110002)	7
462	(000005)	all	1302	(100004)	5	2646	(000022)	$\neq$5
469	(000110)	2	1323	(100101)	$\neq$2,5	2646	(000201)	$\neq$3,5
483	(100011)	3	1386	(100004)	$\neq$5	2800	(110002)	$\neq$3,7

6.11. *Case A_7, $M = 3000$*

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000)	all	658	(0100010)	2	1592	(0010002)	7
8	(0000001)	all	719	(0100010)	7	1624	(0000021)	3
28	(0000010)	all	720	(0100010)	$\neq$2,3,7	1632	(0010100)	2
36	(0000002)	all	784	(0000013)	5	1652	(0000022)	5
56	(0000100)	all	784	(0000200)	3	1680	(0000021)	$\neq$3
62	(1000001)	2	792	(0000005)	all	1708	(0001002)	3
63	(1000001)	$\neq$2	860	(1000011)	3	1716	(0000006)	all
70	(0001000)	all	888	(1000003)	5	1763	(0002000)	5
112	(0000011)	3	924	(1000003)	$\neq$5	1764	(0002000)	$\neq$3,5
120	(0000003)	all	945	(0100002)	all	1800	(0010002)	$\neq$7
168	(0000011)	$\neq$3	952	(0000110)	2	1848	(0000013)	$\neq$5
208	(0100001)	7	1008	(0000110)	$\neq$2,3	2016	(0200001)	3
216	(0100001)	$\neq$7	1016	(0001100)	3	2100	(0001002)	$\neq$3
266	(0000020)	3	1064	(0000102)	5	2128	(1000101)	2
272	(1000002)	3	1092	(0001010)	2	2136	(0001100)	2
280	(1000002)	$\neq$3	1107	(0002000)	3	2289	(0010100)	5
308	(0000101)	2	1128	(0010010)	5	2344	(0001100)	5
330	(0000004)	all	1169	(2000002)	5	2352	(0001100)	$\neq$2,3,5
336	(0000020)	$\neq$3	1176	(0000200)	$\neq$3	2352	(0010100)	$\neq$2,5
378	(0000101)	$\neq$2	1231	(2000002)	3	2400	(1000004)	11
392	(0010001)	2	1232	(2000002)	$\neq$3,5	2464	(0000111)	3
392	(0010001)	3	1244	(1000011)	7	2520	(0000030)	all
420	(0010001)	$\neq$2,3	1280	(1000011)	$\neq$3,7	2520	(0200001)	$\neq$3
448	(0001001)	5	1336	(0010010)	3	2520	(1000004)	$\neq$11
504	(0000110)	3	1344	(0010010)	$\neq$3,5	2584	(0100011)	3
504	(0001001)	$\neq$5	1484	(0001010)	5	2632	(1000101)	3
630	(0000012)	all	1512	(0000102)	$\neq$5	2800	(1000101)	$\neq$2,3
657	(0100010)	3	1512	(0001010)	$\neq$2,5	2828	(1001001)	5

6.12. *Case A_8, $M = 4000$*

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000)	all	966	(00010001)	2	2079	(10000011)	$\neq$2,3,5
9	(00000001)	all	966	(00010001)	3	2304	(00001100)	3
36	(00000010)	all	990	(00000012)	all	2352	(00001010)	2
45	(00000002)	all	1008	(00001001)	$\neq$5	2385	(00100010)	3
79	(10000001)	3	1050	(00010001)	$\neq$2,3	2394	(00100010)	2
80	(10000001)	$\neq$3	1135	(01000010)	7	2520	(00000200)	$\neq$3
84	(00000100)	all	1214	(01000010)	2	2691	(00100010)	7
126	(00001000)	all	1215	(01000010)	$\neq$2,7	2700	(00100010)	$\neq$2,3,7
156	(00000011)	3	1278	(00000013)	5	2772	(00000102)	$\neq$5
165	(00000003)	all	1287	(00000005)	all	2844	(00000021)	3
240	(00000011)	$\neq$3	1359	(10000011)	3	2907	(00002000)	3
306	(01000001)	2	1395	(10000003)	11	2922	(00000022)	5
315	(01000001)	$\neq$2	1440	(10000003)	$\neq$11	2970	(00000021)	$\neq$3
387	(10000002)	5	1461	(01000002)	3	3003	(00000006)	all
396	(10000002)	$\neq$5	1540	(01000002)	$\neq$3	3060	(00010010)	5
414	(00000020)	3	1554	(00000200)	3	3139	(00011000)	3
495	(00000004)	all	1764	(00000110)	2	3168	(00000013)	$\neq$5
504	(00000101)	2	1864	(20000002)	11	3318	(00001010)	5
540	(00000020)	$\neq$3	1890	(00000102)	5	3402	(00001010)	$\neq$2,5
630	(00000101)	$\neq$2	1890	(00000110)	$\neq$2,3	3414	(02000001)	3
684	(00100001)	7	1943	(20000002)	5	3465	(00100002)	all
720	(00100001)	$\neq$7	1944	(20000002)	$\neq$5,11	3654	(00001002)	3
882	(00000110)	3	2034	(10000011)	2	3744	(00010010)	3
882	(00001001)	5	2043	(10000011)	5	3780	(00010010)	$\neq$3,5

6.13. *Case* A_9, $M = 6000$
(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000000)	all	1110	(001000001)	2	2924	(200000002)	11
10	(000000001)	all	1155	(001000001)	$\neq$2	2925	(200000002)	$\neq$3,11
45	(000000010)	all	1452	(000000110)	3	3048	(000000110)	2
55	(000000002)	all	1485	(000000012)	all	3155	(100000011)	11
98	(100000001)	2	1596	(000001001)	5	3156	(000000102)	5
98	(100000001)	5	1826	(010000010)	2	3200	(100000011)	$\neq$3,11
99	(100000001)	$\neq$2,5	1848	(000001001)	$\neq$5	3300	(000000110)	$\neq$2,3
120	(000000100)	all	1860	(000100001)	7	4510	(001000010)	7
210	(000000011)	3	1924	(010000010)	3	4620	(000001010)	2
210	(000001000)	all	1925	(010000010)	$\neq$2,3	4698	(000000021)	3
220	(000000003)	all	1980	(000100001)	$\neq$7	4740	(000001100)	3
252	(000010000)	all	1990	(100000011)	3	4752	(000000102)	$\neq$5
330	(000000011)	$\neq$3	1992	(000000013)	5	4905	(000000022)	5
430	(010000001)	3	2002	(000000005)	all	4940	(001000010)	2
440	(010000001)	$\neq$3	2100	(000010001)	2	4950	(000000021)	$\neq$3
530	(100000002)	11	2100	(000010001)	3	4950	(000000200)	$\neq$3
540	(100000002)	$\neq$11	2145	(100000003)	all	4950	(001000010)	$\neq$2,7
615	(000000020)	3	2278	(010000002)	5	5005	(000000006)	all
715	(000000004)	all	2310	(000010001)	$\neq$2,3	5148	(000000013)	$\neq$5
780	(000000101)	2	2376	(010000002)	$\neq$5	5730	(001000002)	3
825	(000000020)	$\neq$3	2826	(200000002)	3	5940	(020000001)	3
990	(000000101)	$\neq$2	2850	(000000200)	3			

6.14. *Case* A_{10}, $M = 10000$
(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000000)	all	1760	(0010000001)	$\neq$3	4620	(0000010001)	$\neq$2,3
11	(0000000001)	all	2145	(0000000012)	all	4653	(1000000011)	2
55	(0000000010)	all	2277	(0000000110)	3	4653	(1000000011)	5
66	(0000000002)	all	2706	(0000001001)	5	4719	(1000000011)	$\neq$2,3,5
119	(1000000001)	11	2784	(0100000010)	3	4752	(0000100001)	$\neq$7
120	(1000000001)	$\neq$11	2903	(0100000010)	5	4917	(0000000200)	3
165	(0000000100)	all	2904	(0100000010)	$\neq$3,5	4983	(0000000110)	2
275	(0000000011)	3	2959	(1000000011)	3	5016	(0000000102)	5
286	(0000000003)	all	2992	(0000000013)	5	5445	(0000000110)	$\neq$2,3
330	(0000001000)	all	3003	(0000000005)	all	7403	(0000000021)	3
440	(0000000011)	$\neq$3	3014	(1000000003)	13	7722	(0000000102)	$\neq$5
462	(0000010000)	all	3080	(1000000003)	$\neq$13	7865	(0000000021)	$\neq$3
583	(0100000001)	2	3168	(0000001001)	$\neq$5	7876	(0010000010)	2
583	(0100000001)	5	3300	(0001000001)	2	7887	(0000000022)	5
594	(0100000001)	$\neq$2,5	3391	(0100000002)	11	8008	(0000000006)	all
704	(1000000002)	3	3465	(0001000001)	$\neq$2	8008	(0000000013)	$\neq$5
715	(1000000002)	$\neq$3	3510	(0100000002)	$\neq$11	8448	(0000001010)	2
880	(0000000020)	3	4115	(2000000002)	13	8459	(0010000010)	3
1001	(0000000004)	all	4158	(0000010001)	2	8470	(0010000010)	$\neq$2,3
1155	(0000000101)	2	4158	(0000010001)	3	9042	(0000001100)	3
1210	(0000000020)	$\neq$3	4234	(2000000002)	3	9075	(0000000200)	$\neq$3
1485	(0000000101)	$\neq$2	4235	(2000000002)	$\neq$3,13	9405	(0200000001)	3
1705	(0010000001)	3	4422	(0000100001)	7	9713	(0010000002)	5

6.15. *Case* A_{11}, $M = 12000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000000)	all	1716	(00000000020)	$\neq 3$	5720	(00010000001)	$\neq 3$
12	(00000000001)	all	2145	(00000000101)	$\neq 2$	5797	(20000000002)	7
66	(00000000010)	all	2508	(00100000001)	2	5939	(20000000002)	13
78	(00000000002)	all	2508	(00100000001)	5	5940	(20000000002)	$\neq 7,13$
142	(10000000001)	2	2574	(00100000001)	$\neq 2,5$	6642	(10000000011)	11
142	(10000000001)	3	3003	(00000000012)	all	6654	(10000000011)	13
143	(10000000001)	$\neq 2,3$	3432	(00000000110)	3	6720	(10000000011)	$\neq 3,11,13$
220	(00000000100)	all	4069	(01000000010)	5	7656	(00000000102)	5
352	(00000000011)	3	4070	(01000000010)	2	7656	(00000010001)	2
364	(00000000003)	all	4146	(10000000011)	3	7656	(00000010001)	3
495	(00000001000)	all	4211	(01000000010)	11	7788	(00000000110)	2
572	(00000000011)	$\neq 3$	4212	(01000000010)	$\neq 2,5,11$	8074	(00000000200)	3
768	(01000000001)	11	4212	(10000000003)	7	8514	(00001000001)	2
780	(01000000001)	$\neq 11$	4290	(10000000003)	$\neq 7$	8580	(00000000110)	$\neq 2,3$
792	(00000010000)	all	4356	(00000000013)	5	8580	(00000010001)	$\neq 2,3$
912	(10000000002)	13	4356	(00000001001)	5	9009	(00001000001)	$\neq 2$
924	(00000100000)	all	4368	(00000000005)	all	9504	(00000100001)	7
924	(10000000002)	$\neq 13$	4863	(01000000002)	3	10296	(00000100001)	$\neq 7$
1221	(00000000020)	3	5005	(01000000002)	$\neq 3$	11220	(00000000021)	3
1365	(00000000004)	all	5148	(00000001001)	$\neq 5$			
1650	(00000000101)	2	5500	(00010000001)	3			

6.16. *Case* A_{12}, $M = 3000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000000000)	all	455	(000000000003)	all	1287	(000000010000)	all
13	(000000000001)	all	715	(000000001000)	all	1651	(000000000020)	3
78	(000000000010)	all	728	(000000000011)	$\neq 3$	1716	(000000100000)	all
91	(000000000002)	all	988	(010000000001)	2	1820	(000000000004)	all
167	(100000000001)	13	988	(010000000001)	3	2288	(000000000101)	2
168	(100000000001)	$\neq 13$	1001	(010000000001)	$\neq 2,3$	2366	(000000000020)	$\neq 3$
286	(000000000100)	all	1157	(100000000002)	7			
442	(000000000011)	3	1170	(100000000002)	$\neq 7$			

6.17. *Case* A_{13}, $M = 3000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000000000)	all	364	(0000000000100)	all	1442	(1000000000002)	3
14	(0000000000001)	all	546	(0000000000011)	3	1442	(1000000000002)	5
91	(0000000000010)	all	560	(0000000000003)	all	1456	(1000000000002)	$\neq 3,5$
105	(0000000000002)	all	910	(0000000000011)	$\neq 3$	2002	(0000000010000)	all
194	(1000000000001)	2	1001	(0000000001000)	all	2184	(0000000000020)	3
194	(1000000000001)	7	1246	(0100000000001)	13	2380	(0000000000004)	all
195	(1000000000001)	$\neq 2,7$	1260	(0100000000001)	$\neq 13$			

6.18. *Case* A_{14}, $M = 3000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000000000)	all	224	(10000000000001)	$\neq$3,5	1545	(01000000000001)	2
15	(00000000000001)	all	455	(00000000000100)	all	1545	(01000000000001)	7
105	(00000000000010)	all	665	(00000000000011)	3	1560	(01000000000001)	$\neq$2,7
120	(00000000000002)	all	680	(00000000000003)	all	1785	(10000000000002)	all
223	(10000000000001)	3	1120	(00000000000011)	$\neq$3	2835	(00000000000020)	3
223	(10000000000001)	5	1365	(00000000001000)	all			

6.19. *Case* A_{15}, $M = 3000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000000000000)	all	560	(000000000000100)	all	1888	(010000000000001)	5
16	(000000000000001)	all	800	(000000000000011)	3	1904	(010000000000001)	$\neq$3,5
120	(000000000000010)	all	816	(000000000000003)	all	2144	(100000000000002)	17
136	(000000000000002)	all	1360	(000000000000011)	$\neq$3	2160	(100000000000002)	$\neq$17
254	(100000000000001)	2	1820	(000000000001000)	all			
255	(100000000000001)	$\neq$2	1888	(010000000000001)	3			

6.20. *Case* A_{16}, $M = 3000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000000000000)	all	288	(1000000000000001)	$\neq$17	2278	(0100000000000001)	2
17	(0000000000000001)	all	680	(0000000000000100)	all	2295	(0100000000000001)	$\neq$2
136	(0000000000000010)	all	952	(0000000000000011)	3	2380	(0000000000001000)	all
153	(0000000000000002)	all	969	(0000000000000003)	all	2567	(1000000000000002)	3
287	(1000000000000001)	17	1632	(0000000000000011)	$\neq$3	2584	(1000000000000002)	$\neq$3

6.21. *Case* A_{17}, $M = 3000$

(Recall the remark in 6.5.)

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000000000000)	all	322	(10000000000000001)	3	1938	(00000000000000011)	$\neq$3
18	(00000000000000001)	all	323	(10000000000000001)	$\neq$2,3	2718	(01000000000000001)	17
153	(00000000000000010)	all	816	(00000000000000100)	all	2736	(01000000000000001)	$\neq$17
171	(00000000000000002)	all	1122	(00000000000000011)	3			
322	(10000000000000001)	2	1140	(00000000000000003)	all			

6.22. *Case B_2, $M = 300$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00)	all	81	(22)	$\neq 5,7$	199	(07)	11
4	(01)	2	84	(15)	13	199	(25)	7
4	(10)	all	84	(60)	all	200	(71)	11
5	(01)	$\neq 2$	85	(06)	13	204	(07)	$\neq 11,13$
10	(20)	all	86	(23)	5	204	(33)	5
12	(11)	5	91	(05)	$\neq 11$	206	(24)	7
13	(02)	5	105	(41)	all	220	(42)	$\neq 5,7$
14	(02)	$\neq 5$	115	(42)	5	220	(90)	all
16	(11)	$\neq 5$	116	(33)	11	224	(15)	$\neq 11,13$
20	(30)	all	116	(51)	7	231	(61)	all
24	(12)	7	120	(70)	all	236	(53)	13
25	(03)	7	126	(06)	11	251	(25)	13
25	(21)	3	140	(06)	$\neq 11,13$	256	(16)	13
30	(03)	$\neq 7$	140	(14)	$\neq 11$	256	(33)	$\neq 5,7,11$
35	(21)	$\neq 3$	140	(32)	all	260	(24)	$\neq 7,11$
35	(40)	all	144	(17)	17	260	(44)	7
40	(12)	$\neq 7$	145	(08)	17	264	(1,10)	23
44	(31)	7	149	(42)	7	265	(0,11)	23
52	(31)	5	154	(23)	$\neq 5$	271	(08)	13
54	(04)	7	160	(51)	$\neq 7$	284	(08)	11
55	(04)	$\neq 7$	164	(34)	13	284	(36)	17
56	(50)	all	164	(52)	11	285	(08)	$\neq 11,13,17$
60	(14)	11	165	(80)	all	285	(43)	11
61	(05)	11	174	(07)	13	286	(10,0)	all
64	(31)	$\neq 5,7$	179	(24)	11	294	(09)	17
68	(22)	5	180	(18)	19	296	(72)	13
71	(22)	7	180	(33)	7	300	(52)	7
76	(13)	7	181	(09)	19			
80	(13)	$\neq 7$	184	(15)	11			

6.23. *Case B_3, $M = 700$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(000)	all	141	(020)	5	384	(111)	3
6	(001)	2	155	(004)	11	448	(103)	$\neq 11$
7	(001)	$\neq 2$	168	(020)	$\neq 3,5$	448	(202)	5
8	(100)	all	168	(102)	$\neq 3$	448	(310)	7
14	(010)	2	168	(201)	5	472	(111)	7
21	(010)	$\neq 2$	182	(004)	$\neq 11$	483	(022)	5
26	(002)	7	189	(201)	$\neq 5$	512	(111)	$\neq 3,5,7$
27	(002)	$\neq 7$	189	(210)	3	518	(202)	3
35	(200)	all	248	(120)	7	521	(030)	7
40	(101)	7	280	(103)	11	560	(104)	13
48	(101)	$\neq 7$	293	(400)	5	560	(301)	all
63	(011)	3	294	(400)	$\neq 5$	560	(500)	7
64	(011)	2	301	(005)	13	616	(021)	5
64	(110)	5	304	(012)	7	616	(202)	$\neq 3,5$
77	(003)	all	309	(012)	3	664	(310)	5
104	(110)	3	330	(012)	$\neq 3,7$	672	(021)	3
104	(300)	5	344	(111)	5	672	(120)	3
105	(011)	$\neq 2,3$	371	(005)	11	672	(500)	$\neq 7$
112	(110)	$\neq 3,5$	371	(013)	5	687	(006)	13
112	(300)	$\neq 5$	371	(210)	5	693	(021)	$\neq 3,5$
120	(102)	3	378	(005)	$\neq 11,13$			
132	(020)	3	378	(210)	$\neq 3,5$			

6.24. *Case B_4, $M = 1000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000)	all	160	(0011)	2	576	(1002)	≠11
8	(0001)	2	231	(0011)	≠2,3	579	(0110)	3
9	(0001)	≠2	246	(0101)	2	594	(0101)	≠2,7
16	(1000)	all	304	(1010)	7	656	(1100)	3
26	(0010)	2	336	(1100)	5	672	(3000)	≠5
36	(0010)	≠2	369	(0020)	3	752	(1100)	7
43	(0002)	3	406	(0004)	13	768	(1100)	≠3,5,7
44	(0002)	≠3	416	(1010)	2	784	(0110)	2
48	(0100)	2	432	(1010)	≠2,7	798	(2001)	5
84	(0100)	≠2	448	(1002)	11	840	(2001)	3
112	(1001)	3	449	(0004)	11	867	(0012)	3
126	(2000)	all	450	(0004)	≠11,13	874	(0012)	11
128	(1001)	≠3	451	(0020)	7	910	(0012)	≠3,11
147	(0003)	11	495	(0020)	≠3,7	924	(2001)	≠3,5
147	(0011)	3	544	(3000)	5	957	(0013)	5
156	(0003)	≠11	558	(0101)	7			

6.25. *Case B_5, $M = 2000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000)	all	288	(10001)	11	1144	(00020)	≠3,5
10	(00001)	2	320	(00011)	2	1375	(00101)	3
11	(00001)	≠2	320	(10001)	≠11	1376	(10010)	5
32	(10000)	all	330	(01000)	≠2	1408	(01001)	2
44	(00010)	2	418	(00011)	5	1408	(10010)	≠3,5
55	(00010)	≠2	429	(00011)	≠2,3,5	1430	(00101)	≠2,3
64	(00002)	11	462	(20000)	all	1440	(10002)	13
65	(00002)	≠11	670	(00101)	2	1760	(10002)	≠13
100	(00100)	2	749	(00020)	3	1760	(11000)	5
164	(01000)	2	870	(00004)	5	1961	(00012)	11
165	(00100)	≠2	934	(00004)	13	1970	(00012)	13
264	(00003)	13	935	(00004)	≠5,13	1991	(00110)	3
264	(00011)	3	1088	(10010)	3			
275	(00003)	≠13	1143	(00020)	5			

6.26. *Case B_6, $M = 4000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000)	all	442	(000003)	≠5	1729	(000004)	≠5,17
12	(000001)	2	560	(000011)	2	2185	(000020)	11
13	(000001)	≠2	560	(010000)	2	2275	(000020)	≠3,11
64	(000010)	2	704	(100001)	13	2847	(000101)	11
64	(100000)	all	715	(000011)	≠2,3	2925	(000101)	≠2,11
78	(000010)	≠2	715	(001000)	≠2	3392	(100010)	11
89	(000002)	13	768	(100001)	≠13	3808	(001001)	2
90	(000002)	≠13	1287	(010000)	≠2	3838	(000012)	13
208	(000100)	2	1508	(000101)	2	3849	(000012)	3
286	(000100)	≠2	1559	(000020)	3	3849	(000012)	5
364	(001000)	2	1639	(000004)	17	3927	(000012)	≠3,5,13
416	(000011)	3	1716	(200000)	all			
429	(000003)	5	1728	(000004)	5			

6.27. *Case B_7, $M = 5000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000)	all	650	(0000003)	17	1792	(1000001)	$\neq$3,5
14	(0000001)	2	650	(0000011)	3	1912	(0100000)	2
15	(0000001)	$\neq$2	665	(0000003)	$\neq$17	2715	(0000020)	3
90	(0000010)	2	896	(0000011)	2	2821	(0000004)	19
105	(0000010)	$\neq$2	910	(0001000)	2	2884	(0000101)	2
118	(0000002)	3	1090	(0000011)	7	2939	(0000004)	17
118	(0000002)	5	1105	(0000011)	$\neq$2,3,7	2940	(0000004)	$\neq$17,19
119	(0000002)	$\neq$3,5	1288	(0010000)	2	3003	(0010000)	$\neq$2
128	(1000000)	all	1365	(0001000)	$\neq$2	3961	(0000020)	13
336	(0000100)	2	1664	(1000001)	3	4079	(0000020)	7
455	(0000100)	$\neq$2	1664	(1000001)	5	4080	(0000020)	$\neq$3,7,13

6.28. *Case B_8, $M = 7000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000)	all	935	(00000003)	19	4251	(00000020)	3
16	(00000001)	2	935	(00000011)	3	4488	(00100000)	2
17	(00000001)	$\neq$2	952	(00000003)	$\neq$19	4540	(00000004)	7
118	(00000010)	2	1344	(00000011)	2	4691	(00000004)	19
136	(00000010)	$\neq$2	1582	(00001000)	2	4692	(00000004)	$\neq$7,19
151	(00000002)	17	1615	(00000011)	$\neq$2,3	5066	(00000101)	2
152	(00000002)	$\neq$17	2380	(00001000)	$\neq$2	6188	(00010000)	$\neq$2
256	(10000000)	all	3808	(00010000)	2	6528	(01000000)	2
544	(00000100)	2	3840	(10000001)	17	6631	(00000020)	5
680	(00000100)	$\neq$2	4096	(10000001)	$\neq$17	6783	(00000020)	$\neq$3,5

6.29. *Case B_9, $M = 8000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000000)	all	780	(000000100)	2	3876	(000001000)	$\neq$2
18	(000000001)	2	969	(000000100)	$\neq$2	6763	(000000020)	3
19	(000000001)	$\neq$2	1273	(000000011)	3	6936	(000000004)	23
152	(000000010)	2	1292	(000000003)	7	6972	(000010000)	2
171	(000000010)	$\neq$2	1311	(000000003)	$\neq$7	7124	(000000004)	7
188	(000000002)	19	1920	(000000011)	2	7125	(000000004)	$\neq$7,23
189	(000000002)	$\neq$19	2261	(000000011)	$\neq$2,3			
512	(100000000)	all	2906	(000001000)	2			

6.30. *Case B_{10}, $M = 10000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000000)	all	230	(0000000002)	$\neq$3,7	2640	(0000000011)	2
20	(0000000001)	2	1024	(1000000000)	all	3038	(0000000011)	5
21	(0000000001)	$\neq$2	1120	(0000000100)	2	3059	(0000000011)	$\neq$2,3,5
188	(0000000010)	2	1330	(0000000100)	$\neq$2	4466	(0000001000)	2
210	(0000000010)	$\neq$2	1729	(0000000003)	23	5985	(0000001000)	$\neq$2
229	(0000000002)	3	1729	(0000000011)	3	9954	(0000000020)	3
229	(0000000002)	7	1750	(0000000003)	$\neq$23			

6.31. *Case B_{11}, $M = 12000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000000)	all	275	(00000000002)	$\neq$23	2277	(00000000003)	$\neq$5
22	(00000000001)	2	1496	(00000000100)	2	3520	(00000000011)	2
23	(00000000001)	$\neq$2	1771	(00000000100)	$\neq$2	4002	(00000000011)	11
230	(00000000010)	2	2048	(10000000000)	all	4025	(00000000011)	$\neq$2,3,11
253	(00000000010)	$\neq$2	2254	(00000000003)	5	7084	(00000001000)	2
274	(00000000002)	23	2254	(00000000011)	3	8855	(00000001000)	$\neq$2

6.32. *Case* C_3, $M = 1000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(000)	all	172	(300)	7	594	(210)	$\neq$5,7
6	(001)	all	189	(012)	all	616	(120)	$\neq$3,5
8	(100)	2	202	(102)	5	623	(202)	3
13	(010)	3	216	(102)	$\neq$5,7	665	(410)	11
14	(010)	$\neq$3	246	(013)	5	666	(031)	5
14	(100)	$\neq$2	252	(005)	all	666	(500)	11
21	(002)	all	286	(021)	3	786	(015)	7
48	(101)	2	295	(030)	5	792	(007)	all
50	(011)	3	309	(111)	5	798	(014)	5
56	(003)	all	316	(120)	3	813	(301)	5
57	(101)	3	316	(201)	5	840	(130)	7
58	(011)	7	330	(300)	$\neq$7	854	(211)	7
62	(110)	5	350	(021)	$\neq$3	861	(301)	7
63	(200)	5	358	(111)	3	875	(400)	7
64	(011)	$\neq$3,7	378	(201)	$\neq$5	903	(022)	3
70	(101)	$\neq$2,3	385	(030)	$\neq$5	924	(014)	$\neq$5
84	(200)	$\neq$5	423	(111)	7	924	(022)	$\neq$3,5
89	(020)	7	426	(022)	5	924	(112)	5
90	(020)	$\neq$7	448	(013)	$\neq$5	938	(400)	5
112	(110)	2	462	(006)	all	942	(104)	7
126	(004)	all	512	(111)	$\neq$3,5,7	951	(040)	5
126	(110)	$\neq$2,5	525	(103)	all	951	(202)	5
158	(102)	7	552	(120)	5	994	(031)	11
171	(210)	7	573	(210)	5			

6.33. *Case* C_4, $M = 2000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000)	all	312	(1100)	5	792	(1010)	$\neq$2,3,7
8	(0001)	all	313	(2000)	5	825	(0200)	$\neq$3,7
16	(1000)	2	315	(0101)	$\neq$2,3	944	(0102)	5
26	(0010)	2	330	(0004)	all	1016	(1100)	3
27	(0010)	$\neq$2	416	(1010)	2	1048	(1100)	7
36	(0002)	all	504	(0110)	3	1056	(1100)	$\neq$2,3,5,7
40	(0100)	3	513	(1010)	3	1072	(0102)	3
41	(1000)	3	558	(0012)	5	1114	(1002)	3
42	(1000)	$\neq$2,3	593	(2000)	7	1155	(1002)	$\neq$3
48	(0100)	$\neq$3	594	(0012)	$\neq$5	1200	(2100)	7
112	(0011)	3	594	(2000)	$\neq$5,7	1201	(3000)	7
120	(0003)	all	632	(0110)	7	1232	(0102)	$\neq$3,5
128	(1001)	2	744	(0110)	5	1352	(0021)	11
160	(0011)	$\neq$3	765	(1010)	7	1504	(0021)	5
240	(1001)	7	768	(1100)	2	1512	(0021)	$\neq$5,11
246	(0101)	2	784	(0013)	5	1608	(0013)	11
266	(0020)	3	784	(0110)	2	1652	(0022)	5
279	(0101)	3	784	(0200)	3	1716	(0006)	all
281	(0020)	5	789	(0200)	7	1728	(0013)	$\neq$5,11
288	(1001)	$\neq$2,7	792	(0005)	all	1891	(0111)	3
308	(0020)	$\neq$3,5	792	(0110)	$\neq$2,3,5,7			

6.34. *Case C_5, $M = 2500$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000)	all	210	(00011)	3	1099	(10001)	3
10	(00001)	all	220	(00003)	all	1155	(10001)	$\neq$2,3
32	(10000)	2	310	(00011)	11	1276	(01001)	5
43	(00010)	5	320	(00011)	$\neq$3,11	1375	(00012)	3
44	(00010)	$\neq$5	320	(10001)	2	1408	(01001)	$\neq$3,5
55	(00002)	all	615	(00020)	3	1408	(10010)	2
100	(00100)	2	670	(00101)	2	1430	(00012)	$\neq$3
110	(00100)	$\neq$2	715	(00004)	all	1452	(00110)	3
121	(01000)	3	779	(00020)	11	1562	(11000)	5
122	(10000)	3	780	(00020)	$\neq$3,11	1563	(20000)	5
132	(10000)	$\neq$2,3	848	(00101)	5	1992	(00013)	5
164	(01000)	2	891	(00101)	$\neq$2,5	2002	(00005)	all
165	(01000)	$\neq$2,3	1088	(01001)	3			

6.35. *Case C_6, $M = 4000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000)	all	365	(100000)	3	1649	(000020)	13
12	(000001)	all	428	(001000)	5	1650	(000020)	$\neq$3,7,13
64	(000010)	2	429	(001000)	$\neq$2,5	1924	(000101)	5
64	(000010)	3	429	(100000)	$\neq$2,3	1938	(000101)	3
64	(100000)	2	548	(000011)	13	2002	(000101)	$\neq$2,3,5
65	(000010)	$\neq$2,3	560	(000011)	$\neq$3,13	2847	(000012)	7
78	(000002)	all	560	(010000)	2	2925	(000012)	$\neq$7
196	(000100)	5	572	(010000)	$\neq$2,3	3432	(000110)	3
208	(000100)	$\neq$5	768	(100001)	2	3638	(010001)	3
352	(000011)	3	1221	(000020)	3	3652	(100001)	3
364	(000003)	all	1365	(000004)	all	3796	(001001)	5
364	(001000)	2	1508	(000101)	2	3808	(001001)	2
364	(010000)	3	1585	(000020)	7			

6.36. *Case C_7, $M = 6000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000)	all	882	(0000011)	5	2001	(0100000)	5
14	(0000001)	all	896	(0000011)	$\neq$3,5	2002	(0100000)	$\neq$2,3,5
89	(0000010)	7	909	(0001000)	3	2184	(0000020)	3
90	(0000010)	$\neq$7	910	(0001000)	$\neq$3,5	2380	(0000004)	all
105	(0000002)	all	1093	(0100000)	3	2884	(0000101)	2
128	(1000000)	2	1094	(1000000)	3	3093	(0000020)	5
336	(0000100)	2	1288	(0010000)	2	3094	(0000020)	$\neq$3,5
336	(0000100)	3	1430	(1000000)	$\neq$2,3	3795	(0000101)	3
350	(0000100)	$\neq$2,3	1624	(0010000)	5	3811	(0000101)	7
546	(0000011)	3	1638	(0010000)	$\neq$2,5	3900	(0000101)	$\neq$2,3,7
560	(0000003)	all	1792	(1000001)	2	5355	(0000012)	all
820	(0001000)	5	1912	(0100000)	2			

6.37. *Case C_8, $M = 10000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000)	all	1582	(00001000)	2	4862	(10000000)	$\neq$2,3,5
16	(00000001)	all	1699	(00001000)	7	5066	(00000101)	2
118	(00000010)	2	1700	(00001000)	$\neq$2,3,7	5319	(00000020)	17
119	(00000010)	$\neq$2	3264	(00010000)	5	5320	(00000020)	$\neq$3,17
136	(00000002)	all	3280	(01000000)	3	6069	(00100000)	5
256	(10000000)	2	3281	(10000000)	3	6188	(00100000)	$\neq$2,5
528	(00000100)	7	3620	(00000020)	3	6528	(01000000)	2
544	(00000100)	$\neq$7	3792	(00010000)	3	6749	(00000101)	7
800	(00000011)	3	3808	(00010000)	$\neq$3,5	6885	(00000101)	$\neq$2,7
816	(00000003)	all	3876	(00000004)	all	7056	(01000000)	5
1328	(00000011)	17	4096	(10000001)	2	7072	(01000000)	$\neq$2,3,5
1344	(00000011)	$\neq$3,17	4488	(00100000)	2	8908	(00000012)	3
1581	(00001000)	3	4861	(10000000)	5	9044	(00000012)	$\neq$3

6.38. *Case* C_9, $M = 10000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000000)	all	1902	(000000011)	19	7752	(000010000)	$\neq$2,3,7
18	(000000001)	all	1920	(000000011)	$\neq$3,19	8226	(000000101)	2
151	(000000010)	3	2755	(000001000)	7	8416	(000000020)	5
152	(000000010)	$\neq$3	2906	(000001000)	2	8567	(000000020)	19
171	(000000002)	all	2907	(000001000)	$\neq$2,7	8568	(000000020)	$\neq$3,5,19
512	(100000000)	2	5661	(000000020)	3	9216	(100000001)	2
780	(000000100)	2	5985	(000000004)	all	9841	(010000000)	3
798	(000000100)	$\neq$2	6954	(000010000)	3	9842	(100000000)	3
1122	(000000011)	3	6972	(000010000)	2			
1140	(000000003)	all	7734	(000010000)	7			

6.39. *Case* C_{10}, $M = 10000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000000)	all	1024	(1000000000)	2	2640	(0000000011)	$\neq$3,7
20	(0000000001)	all	1100	(0000000100)	3	4466	(0000001000)	2
188	(0000000010)	2	1120	(0000000100)	$\neq$3	4654	(0000001000)	3
188	(0000000010)	5	1520	(0000000011)	3	4655	(0000001000)	$\neq$2,3
189	(0000000010)	$\neq$2,5	1540	(0000000003)	all	8455	(0000000020)	3
210	(0000000002)	all	2620	(0000000011)	7	8855	(0000000004)	all

6.40. *Case* C_{11}, $M = 12000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000000)	all	1496	(00000000100)	5	3520	(00000000011)	$\neq$3,23
22	(00000000001)	all	1518	(00000000100)	$\neq$2,5	6854	(00000001000)	3
229	(00000000010)	11	2002	(00000000011)	3	7083	(00000001000)	5
230	(00000000010)	$\neq$11	2024	(00000000003)	all	7084	(00000001000)	$\neq$3,5
253	(00000000002)	all	2048	(10000000000)	2			
1496	(00000000100)	2	3498	(00000000011)	23			

6.41. *Case D_4, $M = 2000$*

Here, because of the symmetry of the Dynkin diagram, a weight (a_1, a_2, a_3, a_4) and all those which are got from this by permuting a_1, a_2 and a_4 lead to representations of same degree. But for better readability we include all of these weights in this table.

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000)	all	322	(1101)	3	904	(2101)	5
8	(0001)	all	350	(1101)	≠2,3	1008	(0104)	7
8	(0100)	all	384	(0111)	3	1008	(0401)	7
8	(1000)	all	384	(1011)	3	1008	(1004)	7
26	(0010)	2	384	(1110)	3	1008	(1400)	7
28	(0010)	≠2	518	(0202)	3	1008	(4001)	7
35	(0002)	all	518	(2002)	3	1008	(4100)	7
35	(0200)	all	518	(2200)	3	1144	(1102)	7
35	(2000)	all	539	(0012)	5	1144	(1201)	7
48	(0101)	2	539	(0210)	5	1144	(2101)	7
48	(1001)	2	539	(2010)	5	1176	(0021)	3
48	(1100)	2	560	(0005)	7	1176	(0120)	3
56	(0101)	≠2	560	(0500)	7	1176	(1020)	3
56	(1001)	≠2	560	(5000)	7	1256	(0203)	7
56	(1100)	≠2	567	(0012)	≠5	1256	(0302)	7
104	(0003)	5	567	(0210)	≠5	1256	(2003)	7
104	(0011)	3	567	(2010)	≠5	1256	(2300)	7
104	(0110)	3	664	(0013)	5	1256	(3002)	7
104	(0300)	5	664	(0103)	5	1256	(3200)	7
104	(1010)	3	664	(0301)	5	1296	(1102)	≠5,7
104	(3000)	5	664	(0310)	5	1296	(1201)	≠5,7
112	(0003)	≠5	664	(1003)	5	1296	(2101)	≠5,7
112	(0300)	≠5	664	(1300)	5	1322	(0022)	5
112	(3000)	≠5	664	(3001)	5	1322	(0220)	5
152	(0011)	7	664	(3010)	5	1322	(2020)	5
152	(0110)	7	664	(3100)	5	1351	(0006)	7
152	(1010)	7	672	(0005)	≠7	1351	(0600)	7
160	(0011)	≠3,7	672	(0103)	≠5	1351	(6000)	7
160	(0110)	≠3,7	672	(0301)	≠5	1386	(0006)	≠7
160	(1010)	≠3,7	672	(0500)	≠7	1386	(0600)	≠7
168	(0102)	5	672	(1003)	≠5	1386	(6000)	≠7
168	(0201)	5	672	(1300)	≠5	1400	(0021)	≠3
168	(1002)	5	672	(3001)	≠5	1400	(0120)	≠3
168	(1200)	5	672	(3100)	≠5	1400	(1020)	≠3
168	(2001)	5	672	(5000)	≠7	1568	(0013)	≠5
168	(2100)	5	680	(0111)	5	1568	(0310)	≠5
195	(0020)	3	680	(1011)	5	1568	(3010)	≠5
224	(0102)	≠5	680	(1110)	5	1680	(0104)	≠7
224	(0201)	≠5	784	(0111)	2	1680	(0401)	≠7
224	(1002)	≠5	784	(1011)	2	1680	(1004)	≠7
224	(1200)	≠5	784	(1110)	2	1680	(1400)	≠7
224	(2001)	≠5	805	(0202)	5	1680	(4001)	≠7
224	(2100)	≠5	805	(2002)	5	1680	(4100)	≠7
246	(1101)	2	805	(2200)	5	1841	(1111)	3
293	(0004)	5	840	(0111)	≠2,3,5	1896	(0112)	3
293	(0400)	5	840	(0202)	≠3,5	1896	(0211)	3
293	(4000)	5	840	(1011)	≠2,3,5	1896	(1012)	3
294	(0004)	≠5	840	(1110)	≠2,3,5	1896	(1210)	3
294	(0400)	≠5	840	(2002)	≠3,5	1896	(2011)	3
294	(4000)	≠5	840	(2200)	≠3,5	1896	(2110)	3
299	(0020)	7	904	(1102)	5	1925	(0030)	all
300	(0020)	≠3,7	904	(1201)	5			

6.42. *Case D_5, $M = 3000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000)	all	576	(01002)	3	1424	(12000)	7
10	(00001)	all	576	(10002)	3	1424	(21000)	7
16	(01000)	all	606	(00004)	7	1440	(12000)	≠5,7
16	(10000)	all	656	(01100)	3	1440	(21000)	≠5,7
44	(00010)	2	656	(10100)	3	1476	(11001)	5
45	(00010)	≠2	660	(00004)	≠7	1608	(11001)	7
53	(00002)	5	670	(00101)	2	1676	(00200)	3
54	(00002)	≠5	672	(03000)	≠5	1728	(11001)	≠2,5,7
100	(00100)	2	672	(30000)	≠5	1772	(00005)	7
120	(00100)	≠2	720	(01002)	≠3	1772	(00013)	5
126	(02000)	all	720	(10002)	≠3	1782	(00005)	≠7
126	(20000)	all	770	(00020)	≠3	1920	(01003)	7
128	(01001)	5	840	(02001)	3	1920	(10003)	7
128	(10001)	5	840	(20001)	3	2094	(02010)	5
144	(01001)	≠5	880	(12000)	5	2094	(20010)	5
144	(10001)	≠5	880	(21000)	5	2464	(01011)	3
164	(11000)	2	945	(00101)	≠2	2464	(10011)	3
190	(00011)	3	1050	(02001)	≠3	2640	(01003)	≠7
210	(00003)	all	1050	(20001)	≠3	2640	(10003)	≠7
210	(11000)	≠2	1056	(01100)	7	2650	(00110)	7
320	(00011)	≠3	1056	(10100)	7	2708	(00110)	2
416	(01010)	2	1184	(01100)	2	2719	(04000)	5
416	(10010)	2	1184	(10100)	2	2719	(40000)	5
544	(01010)	3	1200	(01100)	≠2,3,7	2772	(04000)	≠5
544	(03000)	5	1200	(10100)	≠2,3,7	2772	(40000)	≠5
544	(10010)	3	1242	(00110)	3	2836	(00102)	5
544	(30000)	5	1333	(00012)	5	2970	(00110)	≠2,3,7
559	(00020)	3	1341	(00012)	3	2976	(01011)	2
560	(01010)	≠2,3	1386	(00012)	≠3,5	2976	(10011)	2
560	(10010)	≠2,3	1408	(11001)	2			

6.43. *Case D_6, $M = 4000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000)	all	462	(020000)	all	1760	(010002)	7
12	(000001)	all	462	(200000)	all	1760	(100002)	7
32	(010000)	all	495	(001000)	≠2	2013	(000101)	5
32	(100000)	all	548	(000011)	11	2079	(000101)	≠2,5
64	(000010)	2	560	(000011)	≠3,11	2112	(010002)	≠7
66	(000010)	≠2	560	(110000)	2	2112	(100002)	≠7
76	(000002)	3	792	(110000)	≠2	2740	(000110)	3
77	(000002)	≠3	1143	(000020)	3	2784	(000012)	3
208	(000100)	2	1286	(000004)	7	2794	(000012)	7
220	(000100)	≠2	1287	(000004)	≠7	2848	(030000)	5
320	(010001)	2	1376	(010010)	5	2848	(300000)	5
320	(010001)	3	1376	(100010)	5	2860	(000012)	≠3,7
320	(100001)	2	1508	(000101)	2	3200	(010100)	2
320	(100001)	3	1561	(000020)	5	3200	(100100)	2
340	(000003)	7	1637	(000020)	11	3808	(001001)	2
340	(000011)	3	1638	(000020)	≠3,5,11	3960	(020001)	7
352	(000003)	≠7	1696	(010010)	11	3960	(200001)	7
352	(010001)	≠2,3	1696	(100010)	11	3992	(000013)	5
352	(100001)	≠2,3	1728	(010010)	≠5,11			
364	(001000)	2	1728	(100010)	≠5,11			

6.44. *Case D_7, $M = 5000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000)	all	832	(0100001)	$\neq 7$	3003	(1100000)	$\neq 2$
14	(0000001)	all	832	(1000001)	$\neq 7$	3079	(0000020)	13
64	(0100000)	all	882	(0000011)	13	3080	(0000020)	$\neq 3,13$
64	(1000000)	all	896	(0000011)	$\neq 3,13$	3913	(0000101)	3
90	(0000010)	2	910	(0001000)	2	4004	(0000101)	$\neq 2,3$
91	(0000010)	$\neq 2$	1001	(0001000)	$\neq 2$	4096	(0100010)	2
103	(0000002)	7	1288	(0010000)	2	4096	(0100010)	3
104	(0000002)	$\neq 7$	1716	(0200000)	all	4096	(1000010)	2
336	(0000100)	2	1716	(2000000)	all	4096	(1000010)	3
364	(0000100)	$\neq 2$	1912	(1100000)	2	4864	(0100010)	13
532	(0000011)	3	1975	(0000020)	3	4864	(1000010)	13
546	(0000003)	all	2002	(0010000)	$\neq 2$	4928	(0100010)	$\neq 2,3,13$
768	(0100001)	7	2275	(0000004)	all	4928	(1000010)	$\neq 2,3,13$
768	(1000001)	7	2884	(0000101)	2			

6.45. *Case D_8, $M = 10000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000)	all	1582	(00001000)	2	5169	(00000020)	7
16	(00000001)	all	1792	(01000001)	2	5303	(00000020)	5
118	(00000010)	2	1792	(10000001)	2	5304	(00000020)	$\neq 3,5,7$
120	(00000010)	$\neq 2$	1820	(00001000)	$\neq 2$	6435	(02000000)	all
128	(01000000)	all	1920	(01000001)	$\neq 2$	6435	(20000000)	all
128	(10000000)	all	1920	(10000001)	$\neq 2$	6528	(11000000)	2
135	(00000002)	all	3483	(00000020)	3	6900	(00000101)	7
544	(00000100)	2	3605	(00000004)	5	7020	(00000101)	$\neq 2,7$
560	(00000100)	$\neq 2$	3740	(00000004)	$\neq 5$	8008	(00100000)	$\neq 2$
768	(00000011)	3	3808	(00010000)	2	8805	(00000012)	3
800	(00000003)	all	4368	(00010000)	$\neq 2$	8925	(00000012)	$\neq 3$
1328	(00000011)	5	4488	(00100000)	2			
1344	(00000011)	$\neq 3,5$	5066	(00000101)	2			

6.46. *Case D_9, $M = 15000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000000)	all	1122	(000000003)	$\neq 5$	5814	(000000004)	$\neq 5,11$
18	(000000001)	all	1902	(000000011)	17	6972	(000010000)	2
152	(000000010)	2	1920	(000000011)	$\neq 3,17$	8226	(000000101)	2
153	(000000010)	$\neq 2$	2906	(000001000)	2	8549	(000000020)	17
169	(000000002)	3	3060	(000001000)	$\neq 2$	8550	(000000020)	$\neq 3,17$
170	(000000002)	$\neq 3$	4096	(010000001)	3	8568	(000010000)	$\neq 2$
256	(010000000)	all	4096	(100000001)	3	11475	(000000101)	$\neq 2$
256	(100000000)	all	4352	(010000001)	$\neq 3$	14043	(000000012)	3
780	(000000100)	2	4352	(100000001)	$\neq 3$	14059	(000000012)	5
816	(000000100)	$\neq 2$	5490	(000000020)	3	14212	(000000012)	$\neq 3,5$
1104	(000000003)	5	5644	(000000004)	11			
1104	(000000011)	3	5813	(000000004)	5			

6.47. *Case D_{10}, $M = 18000$*

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000000)	all	1500	(0000000011)	3	9216	(1000000001)	2
20	(0000000001)	all	1520	(0000000003)	$\neq 11$	9216	(1000000001)	5
188	(0000000010)	2	2620	(0000000011)	19	9728	(0100000001)	$\neq 2,5$
190	(0000000010)	$\neq 2$	2640	(0000000011)	$\neq 3,19$	9728	(1000000001)	$\neq 2,5$
208	(0000000002)	5	4466	(0000001000)	2	12712	(0000000101)	2
209	(0000000002)	$\neq 5$	4845	(0000001000)	$\neq 2$	13089	(0000000020)	19
512	(0100000000)	all	8036	(0000000020)	3	13090	(0000000020)	$\neq 3,19$
512	(1000000000)	all	8644	(0000000004)	11	14344	(0000010000)	2
1120	(0000000100)	2	8645	(0000000004)	$\neq 11$	15504	(0000010000)	$\neq 2$
1140	(0000000100)	$\neq 2$	9216	(0100000001)	2	17575	(0000000101)	3
1500	(0000000003)	11	9216	(0100000001)	5	17765	(0000000101)	$\neq 2,3$

6.48. *Case* D_{11}, $M = 20000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000000)	all	1496	(00000000100)	2	11912	(00000000020)	3
22	(00000000001)	all	1540	(00000000100)	$\neq 2$	12145	(00000000004)	13
230	(00000000010)	2	1958	(00000000011)	3	12397	(00000000004)	$\neq 13$
231	(00000000010)	$\neq 2$	2002	(00000000003)	all	18744	(00000000101)	2
251	(00000000002)	11	3498	(00000000011)	7	18976	(00000000020)	5
252	(00000000002)	$\neq 11$	3520	(00000000011)	$\neq 3,7$	19227	(00000000020)	7
1024	(01000000000)	all	7084	(00000001000)	2	19228	(00000000020)	$\neq 3,5,7$
1024	(10000000000)	all	7315	(00000001000)	$\neq 2$			

6.49. *Case* G_2, $M = 500$

deg	λ	p	deg	λ	p	deg	λ	p
1	(00)	all	97	(21)	5	286	(21)	$\neq 3,5,7$
6	(01)	2	125	(12)	11	295	(22)	11
7	(01)	$\neq 2$	148	(30)	11	301	(05)	13
7	(10)	3	155	(04)	11	371	(05)	11
14	(10)	$\neq 3$	182	(04)	$\neq 11$	371	(13)	5
26	(02)	7	189	(12)	$\neq 11$	378	(05)	$\neq 11,13$
27	(02)	$\neq 7$	189	(21)	3	434	(13)	11
27	(20)	3	196	(30)	5	448	(13)	$\neq 5,11,13$
38	(11)	7	244	(31)	13	469	(14)	5
49	(11)	3	248	(21)	7	481	(22)	7
64	(11)	$\neq 3,7$	259	(13)	13	483	(22)	5
77	(03)	all	267	(40)	7	489	(40)	13
77	(20)	$\neq 3$	273	(30)	$\neq 5,11$			

6.50. *Case* F_4, $M = 12000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000)	all	755	(2000)	7	5369	(1100)	3
25	(0001)	3	1053	(1001)	$\neq 2$	6396	(0101)	2
26	(0001)	$\neq 3$	1053	(2000)	$\neq 7$	6396	(1010)	2
26	(1000)	2	1222	(0100)	3	6707	(1002)	3
52	(1000)	$\neq 2$	1274	(0100)	$\neq 2,3$	7371	(1010)	3
196	(0010)	3	2404	(0011)	3	8424	(1010)	$\neq 2,3,7$
246	(0010)	2	2651	(0003)	7	9477	(2001)	5
246	(0100)	2	2652	(0003)	$\neq 7$	10829	(1002)	$\neq 3$
273	(0010)	$\neq 2,3$	2991	(0011)	7	11102	(3000)	5
298	(0002)	7	3773	(0011)	13	11739	(0110)	5
323	(0002)	13	4096	(0011)	$\neq 3,7,13$	11907	(0101)	3
324	(0002)	$\neq 7,13$	4096	(1100)	2			
676	(1001)	2	4380	(1010)	7			

6.51. *Case* E_6, $M = 50000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(000000)	all	3003	(300000)	$\neq 5$	19305	(010002)	$\neq 3,7$
27	(000001)	all	5643	(001001)	5	19305	(210000)	$\neq 3,7$
27	(100000)	all	5643	(100010)	5	19305	(400000)	$\neq 7$
77	(010000)	3	5746	(000011)	5	20579	(030000)	5
78	(010000)	$\neq 3$	5746	(101000)	5	23179	(110001)	5
324	(000002)	5	5824	(000011)	$\neq 3,5$	26244	(000101)	2
324	(000010)	2	5824	(101000)	$\neq 3,5$	26244	(100100)	2
324	(001000)	2	5994	(100002)	7	28782	(020001)	3
324	(200000)	5	5994	(200001)	7	28782	(120000)	3
351	(000002)	$\neq 5$	6966	(001001)	2	31668	(110001)	2
351	(000010)	$\neq 2$	6966	(100010)	2	32319	(110001)	3
351	(001000)	$\neq 2$	7371	(001001)	$\neq 2,5$	33021	(000020)	5
351	(200000)	$\neq 5$	7371	(100002)	5	33021	(002000)	5
572	(100001)	2	7371	(100010)	$\neq 2,5$	34099	(110001)	7
572	(100001)	3	7371	(200001)	5	34398	(000020)	$\neq 3,5$
650	(100001)	$\neq 2,3$	7722	(100002)	$\neq 5,7$	34398	(002000)	$\neq 3,5$
1377	(010001)	5	7722	(200001)	$\neq 5,7$	34749	(110001)	$\neq 2,3,5,7$
1377	(110000)	5	9828	(010010)	2	35242	(001010)	2
1701	(010001)	13	9828	(011000)	2	40831	(010100)	3
1701	(110000)	13	11934	(010002)	3	43758	(030000)	$\neq 5$
1702	(000100)	2	11934	(210000)	3	44955	(020001)	5
1728	(010001)	$\neq 5,13$	13311	(000004)	7	44955	(120000)	5
1728	(110000)	$\neq 5,13$	13311	(400000)	7	46332	(020001)	$\neq 3,5$
2404	(000011)	3	15822	(010010)	3	46332	(120000)	$\neq 3,5$
2404	(101000)	3	15822	(011000)	3	46656	(001002)	3
2429	(020000)	13	17550	(010010)	$\neq 2,3$	46656	(200010)	3
2430	(020000)	$\neq 13$	17550	(011000)	$\neq 2,3$	46683	(000012)	3
2771	(000100)	3	18225	(000020)	3	46683	(000012)	5
2925	(000100)	$\neq 2,3$	18225	(002000)	3	46683	(201000)	3
3002	(000003)	5	18954	(010002)	7	46683	(201000)	5
3002	(300000)	5	18954	(210000)	7			
3003	(000003)	$\neq 5$	19305	(000004)	$\neq 7$			

6.52. *Case* E_7, $M = 100000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(0000000)	all	6424	(1000001)	19	27664	(0000100)	$\neq 2,3$
56	(0000001)	all	6480	(1000001)	$\neq 7,19$	30704	(0100001)	2
132	(1000000)	2	7106	(0010000)	2	30779	(0100001)	3
133	(1000000)	$\neq 2$	7370	(2000000)	19	39217	(0100001)	7
856	(0100000)	3	7371	(2000000)	$\neq 5,19$	40755	(0100001)	$\neq 2,3,7$
912	(0100000)	$\neq 3$	8512	(0010000)	3	44592	(0000011)	11
1274	(0000010)	2	8645	(0010000)	$\neq 2,3$	50160	(0000011)	2
1330	(0000002)	3	18752	(0000003)	7	51072	(0000011)	$\neq 2,3,11$
1463	(0000002)	$\neq 3$	21184	(0000100)	2	57608	(1100000)	5
1538	(0000010)	7	24264	(0000003)	11	79704	(1100000)	13
1539	(0000010)	$\neq 2,7$	24264	(0000011)	3	86128	(1100000)	2
5568	(1000001)	7	24320	(0000003)	$\neq 7,11$	86184	(1100000)	$\neq 2,5,13$
5832	(2000000)	5	25896	(0000100)	3			

6.53. *Case* E_8, $M = 100000$

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000)	all	23125	(00000002)	7	30132	(00000010)	3
248	(00000001)	all	26504	(00000010)	2	30132	(00000010)	5
3626	(10000000)	2	26999	(00000002)	31	30380	(00000010)	$\neq 2,3,5$
3875	(10000000)	$\neq 2$	27000	(00000002)	$\neq 7,31$			

References

1. D. J. Benson, *Representations and Cohomology I*, vol. 30 of *Cambridge studies in advanced mathematics* (Cambridge University Press, Cambridge, 1991). 143

2. N. BOURBAKI, *Groupes et algèbres de Lie, Chapitres IV, V, VI*, No 1337 of *Actualités Scientifiques et Industrielles* (Hermann, Paris, 1968). 146

3. N. BURGOYNE, 'Modular representations of some finite groups.' 'Representation Theory of Finite Groups and Related Topics,' (AMS, Providence, 1971), vol. XXI of *Proc. Symp. Pure Math.* pp. 13–17. 137

4. R.W. CARTER, *Simple Groups of Lie Type* (A Wiley-Interscience publication, London, 1972). 138

5. C. W. CURTIS and I. REINER, *Methods of representation theory, vol. I, II* (Wiley, New York, 1981/1987). 144

6. M. GECK, G. HISS, F. LÜBECK, G. MALLE and G. PFEIFFER, 'CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras.' *AAECC* 7 (1996) 175–210. 139, 145

7. P. B. GILKEY and G. M. SEITZ, 'Some representations of exceptional Lie algebras.' *Geometriae Dedicata* 25 (1988) 407–416. 135, 137, 138

8. G. HISS and G. MALLE, 'Low-dimensional representations of quasi-simple groups.' *J. Comput. Math., London Mathematical Society* 4 (2001) 22–63. 136, 144

9. J. E. HUMPHREYS, *Introduction to Lie Algebras and Representation Theory*. (Springer Verlag, New York-Heidelberg-Berlin, 1972). 137, 138, 139, 141, 143

10. J. E. HUMPHREYS, 'Modular representations in finite groups of Lie type.' 'Finite Simple Groups II,' (ed. M. J. COLLINS) (Academic Press, London, 1980) pp. 259–290. 136

11. C. JANSEN, K. LUX, R. PARKER AND R. WILSON *An Atlas of Brauer Characters*. (The Clarendon Press Oxford University Press, New York, 1995). 141

12. J. C. JANTZEN, 'Darstellungen halbeinfacher Gruppen und kontravariante Formen.' *J. Reine Angew. Math.* 290 (1977) 117–141. 135

13. J. C. JANTZEN, *Representations of Algebraic Groups*. No. 131 in Pure and Applied Mathematics (Academic Press, London, 1987). 136, 138, 139

14. P. KLEIDMAN and M. LIEBECK, *The subgroup structure of the finite classical groups*. No. 129 in LMS Lecture Note Series (Cambridge University Press, 1990). 135, 143

15. F. LÜBECK, 'On the computation of elementary divisors of integer matrices.' *Journal of Symbolic Computation* 33 (2002) 57–65. 139

16. F. LÜBECK, 'Small degree representations of finite Chevalley groups in defining characteristic.' *LMS J. Comput. Math.* 4 (2001) 135–169. 135

17. R. V. MOODY and J. PATERA 'Fast recursion formula for weight multiplicities' *Bull. Amer. Math. Soc. (N.S.)* 7 (1982) 237–242. 139

18. A. A. PREMET, 'Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic.' *Math. USSR Sb.* 61 (1988) 167–183. (translated from Russian). 139

19. MARTIN SCHÖNERT and OTHERS, *GAP – Groups, Algorithms, and Programming*. Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 3rd edn., (1993–1997). 139

20. R. STEINBERG, 'Lectures on Chevalley groups.' Yale University, (1967). 146

21. R. STEINBERG, 'Endomorphisms of linear algebraic groups.' No. 80 of AMS memoirs (1968). 137

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