Groups with Small Centralizer Bounds

by Christian Weiß.

1 Definition *Let G be a group and $n \in \mathbb{N}$ a positive integer. G has the* centralizer bound n, *if the following two statements are satisfied:*

(i) For all $x \in G$ is $|G : C_G(x)| \leq n$.

(ii) There exists a $t \in G$ with $|G : C_G(t)| = n$.

Equivalent characterization:

2 Theorem *Let G be a group and $n \in \mathbb{N}$ a positive integer. The following statements are equivalent:*

(i) G has the centralizer bound n;

(ii) Every element in G has at most n different conjugates and there exists an element with exactly n conjugates.

Such groups have been called *BFC-groups*.

B.H.Neumann *Groups with finite classes of conjugate subgroups*, Mathematische Zeitschrift 63, p. 76-96 (1955/56):

BFC-groups are exactly the groups with finite commutator subgroup (*FD-groups*).

"... our methods do not allow us to state a bound for the order of the derived group in a group in which a finite bound is given for the cardinals of the classes of conjugate elements (...).
One is naturally led to conjecture that such bounds for the order of the derived group or the index of the centre do exist".

James Wiegold showed in his article *Groups with boundedly finite classes of conjugate elements*, Proceedings Royal Society London Series A 238 (1957), p. 389-401:

If G is an n-BFC-group (i.e. a group with centralizer bound n), then

- $|G'| \leq n^N$, N of order $\frac{1}{2}\, n^4\, (\log_2 n)^3$.

- $n = p$, where p is a prime:

  $|G'| = p$, G' cyclic, G solvable with derived length 2.

3 Example • *Dihedral group D_p with an odd prime p has centralizer bound p.*

	e	d	d²	s	sd	sd²	I
e	e	d	d²	s	sd	sd²	1
d	d	d²	e	sd²	s	sd	2
d²	d²	e	d	sd	sd²	s	2
s	s	sd	sd²	e	d	d²	3
sd	sd	sd²	s	d²	e	d	3
sd²	sd²	s	sd	d	d²	e	3

- *Dihedral group D_4 has centralizer bound 2.*

- *Every abelian group has centralizer bound 1.*

4 Theorem *Let G be a group and $n \in \mathbb{N}$ a positive integer. Let G have the centralizer bound n.*

(i) Every subgroup and every factor group of G has a centralizer bound of at most n.

(ii) If H is a group with centralizer bound m, then $H \otimes G$ has the centralizer bound $m \cdot n$.

Especially:

(iii) If A is an arbitrary abelian group, then $A \otimes G$ also has the centralizer bound n.

The Aims of my Thesis

Bertram Huppert: *Endliche Gruppen I*, Springer Verlag, Berlin Heidelberg (1967)):

5 Theorem *Let U be a subgroup of the group G of index $|G : U| = n \in \mathbb{N}$. Then we have:*

(i) $D := \bigcap_{x \in G} xUx^{-1}$ is a normal subgroup of G;

(ii) G/D is isomorphic to a transitive subgroup of S_n.

6 Corollary *Let U be a subgroup of the group G of index $|G : U| = n \in \mathbb{N}$. Then there exists a normal subgroup $N \trianglelefteq G$ with the following property:*

$|G/N| = n \cdot m$, where m is a divisor of $(n-1)!$.

$N := \bigcap_{x \in G} xUx^{-1}$ is such a normal subgroup.

Group G has centralizer bound n

$\Longrightarrow$ there exists $t \in G$ such that $|G : C_G(t)| = n$.

Applying the theorems before we get

- $D(t) := \bigcap_{x \in G} xC_G(t)x^{-1}$.

  $G/D(t)$ isomorphic to transitive subgroup of S_n.

- $|G/D(t)| = n \cdot m$, where $m \mid (n-1)!$.

Questions:

- Stronger limitations on the order of $G/D(t)$?

- Properties? (Solvability, nilpotency, ...?)

- Structure? (Commutator subgroup G', center $Z(G)$, $G/Z(G)$, ...?)

Groups with Centralizer bound p

$t \in G$ is element with $|G : C_G(t)| = p$.

$D(t) \; := \; \bigcap_{x \in G} \; x C_G(t) x^{-1}$.

$$G \text{ has centralizer bound } p \quad \Longleftrightarrow \quad |G'| = p.$$

Results:

- $G \; = \; \langle v, w, D(t) \rangle$ with $v^m \in \langle w, D(t) \rangle$ and $m \mid p-1$.

- G has normal subgroups of index $m \cdot p$ and of index m, where $m \mid p - 1$.

Case m=1

$G = \langle w, D(t) \rangle$ and $C_G(t) \equiv D(t) \trianglelefteq G$.

7 Theorem *Let $C_G(t)$ be a normal centralizer of index p in G.*

- *If $x \in G$ arbitrary, then either $x \in Z(G)$ or $|G : C_G(x)| = p$, where $C_G(x) \trianglelefteq G$.*

- *G is nilpotent of class 2.*

- *The factor group $G/Z(G)$ is abelian of exponent p.*

Case $m > 1$

$G = \langle v, w, D(t) \rangle$ with $v^m \in \langle w, D(t) \rangle$, $m \mid p-1$,

and $C_G(t) \ntrianglelefteq G$.

8 Theorem *Let $C_G(t)$ be a non-normal centralizer of index p in G.*

- $D(t) \equiv Z(G)$;
- *G is not nilpotent.*

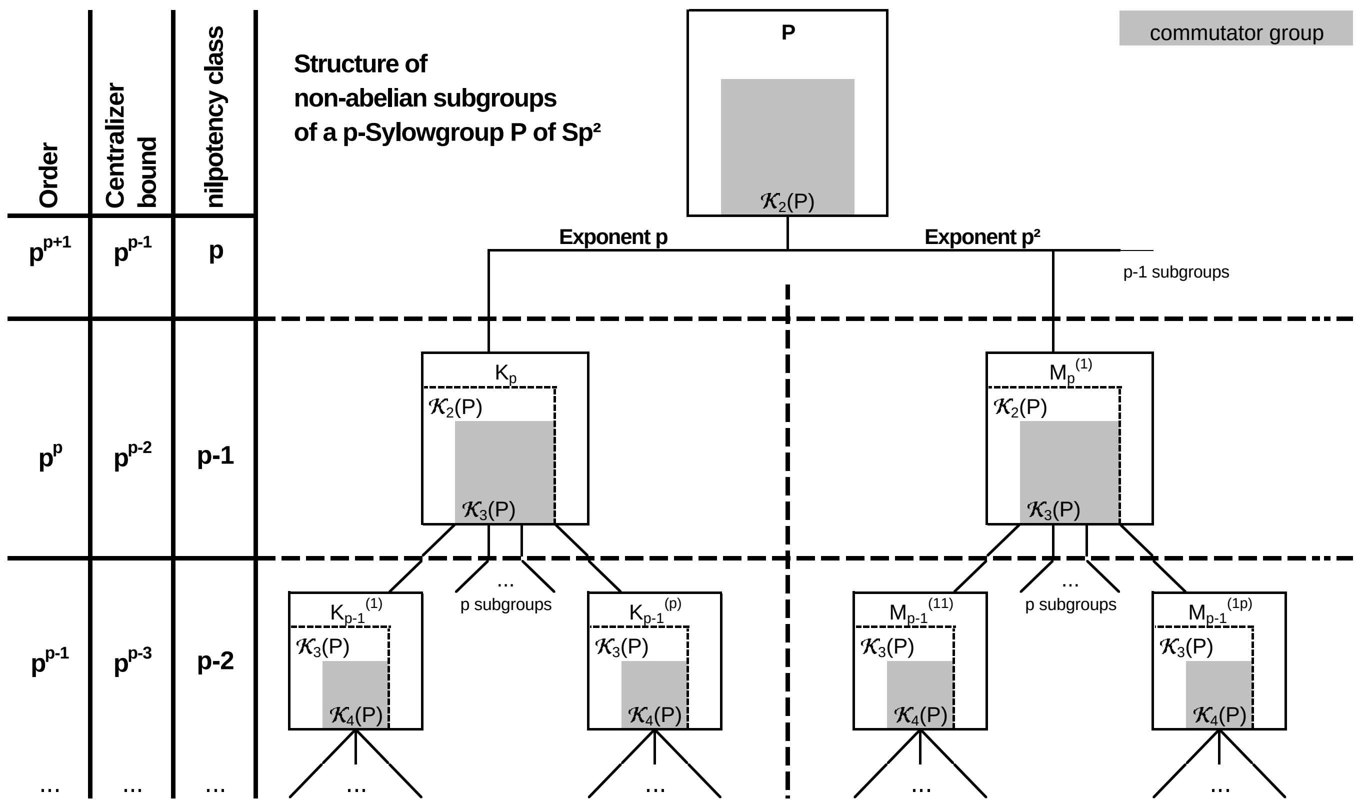
p-Groups with Centralizer Bound p^2

$|G : C_G(t)| = p^2$; $D(t) := \bigcap_{x \in G} xC_G(t)x^{-1}$.

By Theorem 5:

$G/D(t)$ isomorphic to subgroup of p-Sylowgroup of S_{p^2}.

Structure of p-Sylowgroup of S_{p^2}:



Case 1: Every centralizer of index p^2 is normal

Every centralizer of index p^2 is normal

$\Longrightarrow$ Every centralizer is normal

$\Longrightarrow$ $G' \subseteq Z(G)$

$\Longrightarrow$ G is nilpotent of class 2 and solvable with derived length 2!

9 Theorem *Suppose that every centralizer of index p^2 is normal.*

- *The order of G' is either p^2 or p^3;*
- *if $|G'| = p^3$, then G' is of exponent p and also the factor group $G/Z(G)$ is of order p^3 and exponent p.*

10 Example • *$G_1 \times G_2$, where G_1, G_2 are p-groups with centralizer bound p; then $|G'| = p^2$.*

- *$G :=$*
$$\{\begin{pmatrix} 1 & x & y_1 & * & * \\ & 1 & x_2 & y_2 & * \\ & & 1 & x_3 & y_3 \\ & & & 1 & x \\ & & & & 1 \end{pmatrix} \mid x, x_2, x_3, y_1, y_2, y_3 \in \mathbb{F}_p\};$$
we have $G' \equiv Z(G)$ of order p^3 and $|G/Z(G)| = p^3$.

- *$G := \{\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \mid a, b, c \in GF(p^2)\}$;*
we have $G' \equiv Z(G)$ of order p^2 and $|G/Z(G)| = p^4$.

- *$G :=$*
$$\{\begin{pmatrix} 1 & x_1 & x_2 & x_3 & x_4 \\ & 1 & y_1 & y_2 & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid x_1, ..., x_4, y_1, y_2 \in \mathbb{F}_p\};$$
we have $G' \subsetneqq Z(G)$, $|G'| = p^2$ and $|G/Z(G)| = p^3$.

Case 2: There exists a centralizer $C_G(t)$ of index p^2, which is NOT normal

The centralizer $C_G(t)$ of index p^2 is not normal

$\Longrightarrow$ its normalizer is of index p

$\Longrightarrow$ its normalizer is normal

$\Longrightarrow$ G' is contained in its normalizer

$\Longrightarrow$ for all $x, y, z \in G$ we have

$$[x, [x, [y, z]]] \;=\; e,$$

i.e. G' fulfills a 2-Engel condition!

Hermann Heineken: *Engelsche Elemente der Länge 3*. Illinois J. Math. 5 (1961), S. 681-707:

11 Theorem *Let* $[x,[x,[x,y]]] = e$ *for all* $x, y \in G$ *(3-Engel condition) and let* G *contain no elements of order 2 or 5, then* G *is nilpotent of class at most 4.*

But it is possible to find the following stronger result:

12 Theorem *If* G' *fulfills a 2-Engel condition and if* G *contains no elements of order 2, then* G *is nilpotent of class at most 3.*

13 Theorem *Suppose that there exists a centralizer of index p^2 which is not normal.*

- *G is nilpotent of class 3 and solvable with derived length 2.*
- *G' is also of order p^2 or p^3.*
- *If $|G'| = p^3$ ($p \neq 2$), then $G/Z(G)$ is of order p^3.*

14 Example

- *The dihedral group D_8.*
- *Non-abelian subgroups of order p^4 in a p-Sylow-group of S_{p^2}, $p > 2$;*

  we have $|G'| = p^2$ and $|G/Z(G)| = p^3$.
- $G = \{ \begin{pmatrix} 1 & a_1 & b_1 & c_1 & * \\ & 1 & a_2 & b_2 & c_2 \\ & & 1 & -a_1 & b_1 \\ & & & 1 & -a_2 \\ & & & & 1 \end{pmatrix} \mid a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{F}_p \}$;

  we have $|G'| = p^3$ and $|G/Z(G)| = p^3$.

$$
\begin{array}{lcl}
G \text{ has centralizer bound } p^2 & \Longrightarrow & |G'| = p^2 \text{ or } p^3; \\
G \text{ has centralizer bound } p^2 & \Longleftarrow & |G'| = p^2; \\
G \text{ has centralizer bound } p^2 & \not\Longleftarrow & |G'| = p^3.
\end{array}
$$

Counter examples for the last case:

- Non-abelian subgroup of order p^5 in a p-Sylow-group of S_{p^2}, $p \geq 5$.
- The direct product of three p-groups with centralizer bound p.

For further results I refer to my thesis, which can be found on

http://mathephysik.berlios.de/uni.html