Character Tables of Parabolic Subgroups of Steinberg's Triality Groups

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Steinberg's triality groups ${}^3D_4(q)$:

- For each $q = p^n$, $n \in \mathbb{N}_{\geq 1}$, p prime, ${}^3D_4(q)$ is a finite simple twisted Chevalley group of order

$$|{}^3D_4(q)| = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$$

- Ordinary character table of ${}^3D_4(q)$:
  N. Spaltenstein (1982)
  D.I. Deriziotis, G.O. Michler (1987)

- ℓ-modular decomposition matrices of ${}^3D_4(q)$ for q odd, $\ell > 2$ prime, $\ell \not| \, q$:
  M. Geck (1991), up to a few numbers in the principal block

- Idea:
  Ordinary and modular representation theory of parabolic subgroups of ${}^3D_4(q)$ might help to determine the missing decomposition numbers

  (inspired by T. Okuyama's and K. Waki's determination of the decomposition numbers of $\mathrm{Sp}_4(q)$, 1998)

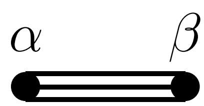
- First Step:
  Computation of the character tables of the parabolic subgroups of ${}^3D_4(q)$

Parabolic subgroups of ${}^3D_4(q)$:

- Associated with each Chevalley group is a root system.

- Root System of ${}^3D_4(q)$: Type G_2

  Dynkin diagram:



- Associated with each subset of the set $\{\overset{\alpha}{\bullet}, \overset{\beta}{\bullet}\}$ of nodes of the Dynkin diagram is a parabolic subgroup of ${}^3D_4(q)$ (up to conjugacy)

  $\{\overset{\alpha}{\bullet}, \overset{\beta}{\bullet}\}$: ${}^3D_4(q)$

  $\{\overset{\alpha}{\bullet}\}$: max. parabolic subgroup P

  $\{\overset{\beta}{\bullet}\}$: max. parabolic subgroup Q

  $\emptyset$: Borel subgroup B

Character tables of B, P, Q:

- for $p > 2$:   H. (2003)

  (inspired by H. Enomoto's and H. Yamada's determination of the character tables of the parabolic subgroups of $G_2(2^n)$, 1986)

Computational tools:

- Programs based on CHEVIE for computing in groups of Lie type (C. Köhler, H.)

- MAPLE–programs based on CHEVIE for restriction and induction of characters between generic character tables.

- Library of generic character tables in CHEVIE : M. Geck, G. Hiss, F. Lübeck, G. Malle, J. Michel, G. Pfeiffer

From now on: $p > 2$.

Character table of B:

- Conjugacy classes of B and their fusions in ${}^3D_4(q)$: M. Geck (1991)
- Have constructed each $\chi \in \mathrm{Irr}(B)$ by inducing linear characters of subgroups of B
- B is an M–group

Character table of Q:

- Have computed the conjugacy classes of Q using the fusions of the classes of B in ${}^3D_4(q)$

<u>Construction of the irred. characters of Q:</u>

(similar for P)

Methods:

1 Clifford theory applied to the Levi decomposition

$$Q = L_Q \ltimes U_Q$$

where $L_Q \cong \mathbb{Z}_{q^3-1} \ltimes SL_2(q)$ and $|U_Q| = q^{11}$

2 Decomposition of restrictions of unipotent characters of ${}^3D_4(q)$ to Q into irreducible constituents

Inertia subgroups in Q

Q acts on Irr(U_Q) by conjugation.

There is a set of representatives $\{\psi_0, \ldots, \psi_6\}$ for the orbits of Q on Irr(U_Q) such that:

- ψ_0 is the trivial character,
- ψ_1, ψ_2 are linear characters,
- $\psi_3, \ldots, \psi_6$ have degree q^3,
- for the inertia groups $I_j :=$ Stab$_Q(\psi_j)$ we have:

j	Structure of I_j/U_Q	I_j
0	L_Q	$= Q$
1	$\mathbb{Z}_{q-1} \ltimes (\mathbb{F}_q, +)$	$\subseteq B$
2	$\{1\}$	$\subseteq B$
3	$SL_2(q)$	$\not\subseteq B$
4	$\mathbb{Z}_{q^3-1} \ltimes (\mathbb{F}_q, +)$	$\subseteq B$
5	$\mathbb{Z}_2 \ltimes (\mathbb{F}_q, +)$	$\subseteq B$
6	$\mathbb{Z}_2 \ltimes (\mathbb{F}_q, +)$	$\subseteq B$

Construction of the irred. characters of Q:

- Construct the irreducible characters of Q not covering ψ_0, ψ_3 by induction from B

- Get the irreducible characters of Q covering ψ_0 by inflating the characters of L_Q

- Construct the remaining irreducible characters by decomposing restrictions of unipotent characters of ${}^3D_4(q)$ to Q into irreducible constituents, orthogonality relations and products of characters