

# Spherical designs and lattices

Elisabeth Nossek

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**Abstract** In this article we prove that integral lattices with minimum  $\leq 7$  (or  $\leq 9$ ) whose set of minimal vectors form spherical 9-designs (or 11-designs respectively) are extremal, even and unimodular. We furthermore show that there does not exist an integral lattice with minimum  $\leq 11$  which yields a 13-design.

**Keywords** extremal lattices · spherical designs · strongly perfect lattices

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## 1 Introduction

The density of a sphere packing associated to a lattice  $\Lambda$  is given through the Hermite function  $\gamma(\Lambda)$ . The local maxima of  $\gamma$  are called extreme lattices and were characterised through the geometry of their shortest vectors,  $S(\Lambda) := \{l \in \Lambda | (l, l) = \min(\Lambda)\}$ , where  $\min(\Lambda) := \min\{(x, x) | 0 \neq x \in \Lambda\}$ , in the works of Voronoi (1908), Korkine and Zolotareff (1872). A prominent subclass of extreme lattices are the strongly perfect lattices introduced by Venkov (2001). They are characterised by the property that  $S(\Lambda)$  forms a spherical 5-design:

**Definition 1** A finite subset  $X$  of the  $n$ -dimensional sphere  $S^{n-1}(m)$  of radius  $m$  forms a spherical  $t$ -design if

$$\int_{S^{n-1}(m)} f(x) dx = \frac{1}{|X|} \sum_{x \in X} f(x)$$

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Elisabeth Nossek  
Lehrstuhl D für Mathematik, RWTH Aachen, Templergraben 64, D-52062 Aachen, Germany  
Tel.: +49-241-80-94554  
Fax: +49-241-80-92108  
E-mail: Elisabeth.Nossek@rwth-aachen.de

for all homogeneous polynomials  $f$  in  $n$  Variables and of degree  $\leq t$ . A lattice  $\Lambda$  such that  $S(\Lambda)$  is a spherical  $t$ -design is called a  $t$ -design lattice.

The classification of strongly perfect lattices is known up to dimension 12 (Nebe and Venkov (2000), Nebe and Venkov (2005)), but becomes very complicated in higher dimensions. Venkov (2001) and Martinet (2001) imposed further design conditions and classified all integral lattices of  $\min \leq 3$  (resp.  $\min \leq 5$ ) whose minimal vectors form spherical 5-designs (resp. 7-designs).

This paper extends their work, more precisely we prove the following theorem:

- Theorem 2**
1. *The only integral 9-design lattices with minimum  $\leq 7$  are the Leech lattice  $\Lambda_{24}$  and the extremal even unimodular lattices in dimension 48.*
  2. *The only integral 11-design lattices with minimum  $\leq 9$  are  $\Lambda_{24}$  and the 48 and 72 dimensional extremal even unimodular lattices.*
  3. *There is no integral 13-design lattice with minimum  $\leq 11$ .*

## 2 Some facts about spherical designs and lattices

As 9 and 11-designs are also 7-designs, we will summarize their classification known from Martinet (2001):

**Theorem 3** *The integral 7-design lattices with minimum  $\leq 5$  are  $\mathbb{E}_8$ , the unimodular lattice  $\mathcal{O}_{23}$  with minimum 3, the three laminated lattices  $\Lambda_{16}$  (the Barnes-Wall lattice),  $\Lambda_{23}$  and  $\Lambda_{24}$  (the Leech lattice) and the unimodular lattices of dimension 32 and minimum 4.*

Martinet also proves that only the Leech lattice is an 11-design lattice and the other lattices in Theorem 3 do not yield 8-designs (Martinet 2001, Proposition). Hence the only integral lattice with minimum  $\leq 5$  whose minimal vectors form a 9 or 11-design is the Leech lattice.

In this article we will use the following characterisation (see (Venkov 2001, th. 3.2)):

**Theorem 4** *A finite set  $X = -X \subset \mathcal{S}^{n-1}(m)$  forms a spherical  $2t+1$ -design if and only if*

$$D_{2i}(\alpha) := \sum_{x \in X} (x, \alpha)^{2i} = c_i |X| m^i(\alpha, \alpha)^i$$

with  $c_i := \prod_{k=0}^{i-1} \frac{1+2k}{n-2k}$

*holds for all  $i \leq t$  and all  $\alpha \in \mathbb{R}^n$ .*

In the following we will often distinguish between unimodular and non-unimodular lattices. If  $\Lambda$  is an integral non-unimodular lattice then for  $v \in \Lambda^*$

minimal in its class modulo  $\Lambda$  holds that  $|(v, \lambda)| \leq \frac{\min(\Lambda)}{2}$  for all  $\lambda \in S(\Lambda)$  ((Martinet 2001, Lemme 1.1)). For even non-unimodular lattices  $\Lambda$  we know that  $\Lambda^*/\Lambda$  is a regular quadratic group in particular there exists an element  $w \in \Lambda^*$  with  $(w, w) \notin 2\mathbb{Z}$  and we can assume w.l.o.g. that such a  $w$  is minimal in its class.

### 3 9-design lattices of minimum $\leq 7$

Throughout this section  $\Lambda \subseteq \mathbb{R}^n$  denotes an integral 9-design lattice of minimum  $m \leq 7$  with  $X \cup -X := S(\Lambda)$  and  $s := |X|$ .

We will start by proving part 1 of Theorem 2. The characterisation in Theorem 4 leads to the following system of linear equations for which only integral solutions correspond to integral 9-design lattices.

**Lemma 5** *For all  $\alpha \in S(\Lambda)$  put  $s_i(\alpha) := |\{x \in X|(x, \alpha) = \pm i\}|$ . The  $s_i$  are independent of  $\alpha$  and  $s_i = 0$  for  $i > 3$ . The following system of linear equations has non-negative integral solutions for the  $s_i$  and for  $s$  if  $S(\Lambda)$  is a spherical 9-design:*

$$\begin{pmatrix} 1 & 2^2 & 3^2 \\ 1 & 2^4 & 3^4 \\ 1 & 2^6 & 3^6 \\ 1 & 2^8 & 3^8 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} \frac{sm^2}{n} - m^2 \\ \frac{3sm^4}{n(n+2)} - m^4 \\ \frac{15sm^6}{n(n+2)(n+4)} - m^6 \\ \frac{105sm^8}{n(n+2)(n+4)(n+6)} - m^8 \end{pmatrix}.$$

*Proof* The system of equations is just a result of the evaluation of the equations in Theorem 4 for  $\alpha \in S(\Lambda)$ .  $\square$

*Remark 6* A simple calculation with Pari shows that for  $m = 7$  there are no non-negative integral solutions  $(n, s, s_1, s_2, s_3) \in \mathbb{Z}_{>0}^5$ . For  $m = 6$  non-negative integral solutions exist only for the following values of  $n$  and  $s$ :

**Table 1** Dimensions and kissing numbers for integral 9-design lattices.

n	26	36	44	46	48	49
s	69888	1149120	8500800	13395200	26208000	50992095

Following a method used in Martinet (2001) we will have a look at non-unimodular lattices at first.

**Lemma 7** *If  $\Lambda$  is non-unimodular and  $\min(\Lambda) = 6$  then  $n \in \{26, 36\}$ .*

*Proof* For all elements  $v \in \Lambda^* \setminus \Lambda$  that are minimal in their class modulo  $\Lambda$  we can define  $t_i(v) := |\{x \in S(\Lambda)|(x, v) = i\}|$ . The  $t_i$  are independent of  $v$  and

for  $i > 4$   $t_i = 0$ . Therefore we get a system of equations again with  $t := (v, v)$ :

$$\begin{pmatrix} 1 & 2^2 & 3^2 \\ 1 & 2^4 & 3^4 \\ 1 & 2^6 & 3^6 \\ 1 & 2^8 & 3^8 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \frac{smt}{n^2 t^2} \\ \frac{3sm^2 t^2}{n(n+2)} \\ \frac{15sm^3 t^3}{n(n+2)(n+4)} \\ \frac{105sm^4 t^4}{n(n+2)(n+4)(n+6)} \end{pmatrix}.$$

$t$  has to be rational and positive. For every pair  $(n, s)$  from Table 1 we get a solution of the system and a polynomial equation  $p_n$  of degree 4 whose positive rational roots are the possible values for  $t$ . But the only cases in which  $p_n$  has such roots are  $n = 26$  where  $t \in \{\frac{8}{3}, 4\}$  and for  $n = 36$  where  $t = 4$ .  $\square$

**Lemma 8** *There is no non-unimodular lattice in dimension 26 or 36 such that its set of minimal vectors form a spherical 9-design.*

*Proof* Let  $\Lambda$  be a non-unimodular lattice. Without loss of generality we can assume that  $\Lambda$  is generated by its minimal vectors, hence  $\Lambda$  is even. For  $n = 36$  we know that  $(v, v) = 4$  for all  $v$  in  $\Lambda^* \setminus \Lambda$  with minimal norm in its class modulo  $\Lambda$ . Hence  $\Lambda^*$  has to be even and therefore unimodular which contradicts our assumption.

For  $n = 26$  we know that  $(v, v) \in \{\frac{8}{3}, 4\}$  for  $v$  in  $\Lambda^* \setminus \Lambda$  with minimal norm in its class modulo  $\Lambda$ .  $\Lambda^*/\Lambda$  is a regular quadratic  $\mathbb{F}_3$  space with  $q : \Lambda^*/\Lambda \rightarrow \mathbb{F}_3$  with  $q(x + \Lambda) := \frac{3(x, x)}{2} \pmod{3}$ . Because  $q(\Lambda^*/\Lambda) = \{0, 1\}$  we know that  $\Lambda^*/\Lambda$  is an one-dimensional  $\mathbb{F}_3$  space with a generator  $v$  with  $q(v) = 1$ . Hence  $\det(\Lambda) = 3$  and  $\gamma(\Lambda) = \frac{6}{3^{1/26}}$  which is greater than the Hermite constant  $\gamma_{26}$  (see (Cohn and Elkies 2003, Table 3)).  $\square$

**Lemma 9** *If  $\Lambda$  is unimodular and  $\min(\Lambda) = 6$  then  $n = 48$  and  $\Lambda$  is even and extremal.*

*Proof* Let  $\Lambda^{(e)} := \{\lambda \in \Lambda | (\lambda, \lambda) \in 2\mathbb{Z}\}$  be the even sublattice of  $\Lambda$  then  $S(\Lambda^{(e)}) = S(\Lambda)$  and  $\Lambda^{(e)}$  is even and unimodular as a result of Lemma 7 and Lemma 8. Therefore  $n$  has to be divisible by 8 as a result of a theorem by Hecke (see e.g. (Koecher and Krieg 1998, Satz V.2.5)), hence  $n = 48$ . As  $\Lambda^{(e)}$  is unimodular it has to be equal to  $\Lambda$ , so  $\Lambda$  is even and obviously extremal.  $\square$

This concludes the proof of Theorem 2 Part 1.

**Corollary 10** *Both the Leech lattice and the 48-dimensional even unimodular lattices yield not only 9-designs but also 11-designs.*

*Proof* These lattice are all even, unimodular and extremal and their dimension is divisible by 24 hence their sets of minimal vectors form 11-designs by a theorem by Venkov (see e.g. (Conway and Sloane 1998, Chapter 7, Theorem 23)).  $\square$

#### 4 11-design lattices with minimum $\leq 9$

Throughout this section  $\Lambda \subseteq \mathbb{R}^n$  denotes an integral 11-design lattice with minimum  $m \leq 9$  and  $s = |X|$  with  $X \cup -X := S(\Lambda)$ . We will proceed in this section with the proof of theorem 2 part 2 and compute the possible values for the dimension and the kissing number in the same way as in Lemma 5.

**Lemma 11** *Using the definitions in the proof of Lemma 5 we get that  $s_i = 0$  for  $i > 4$ . The following system of linear equations has non-negative integral solutions for the  $s_i$  and for  $s := |X|$  if  $S(\Lambda)$  is a spherical 11-design:*

$$\begin{pmatrix} 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^4 & 3^4 & 4^4 \\ 1 & 2^6 & 3^6 & 4^6 \\ 1 & 2^8 & 3^8 & 4^8 \\ 1 & 2^{10} & 3^{10} & 4^{10} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} \frac{sm^2}{n} - m^2 \\ \frac{3sm^4}{n(n+2)} - m^4 \\ \frac{15sm^6}{n(n+2)(n+4)} - m^6 \\ \frac{105sm^8}{n(n+2)(n+4)(n+6)} - m^8 \\ \frac{945sm^{10}}{n(n+2)(n+4)(n+6)(n+8)} - m^{10} \end{pmatrix}.$$

*Remark 12* We get no solutions  $(n, s, s_i)_{i \leq 4} \in \mathbb{Z}_{>0}^6$  for  $m = 9$  and for  $m = 8$  we get such solutions only for the following values of  $n$  and  $s$ :

**Table 2** Dimensions and Kissing numbers for 11-design lattices.

n	50	56	62	64	66
s	57256875	237875400	1071285600	1866110400	3236535225
n	68	72	76	78	82
s	474335190	3109087800	1263241980	866338200	470377215

Now we can see with the same arguments as in Lemma 7 and Lemma 8 that an integral 11-design lattice has to be unimodular.

**Lemma 13** *There is no non-unimodular lattice with minimum 8 whose minimal vectors form a spherical 11-design.*

*Proof* For all elements  $v \in \Lambda^* \setminus \Lambda$  that are minimal in their class modulo  $\Lambda$  we can define  $t_i(v) := |\{x \in S(\Lambda) | (x, v) = i\}|$ . The  $t_i$  are independent of  $v$  and for  $i > 5$   $t_i = 0$ . Therefore we get a system of equations again with  $t := (v, v)$ :

$$\begin{pmatrix} 1 & 2^2 & 3^2 & 4^2 \\ 1 & 2^4 & 3^4 & 4^4 \\ 1 & 2^6 & 3^6 & 4^6 \\ 1 & 2^8 & 3^8 & 4^8 \\ 1 & 2^{10} & 3^{10} & 4^{10} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} \frac{smt}{n} \\ \frac{3sm^2t^2}{n(n+2)} \\ \frac{15sm^3t^3}{n(n+2)(n+4)} \\ \frac{105sm^4t^4}{n(n+2)(n+4)(n+6)} \\ \frac{945sm^5t^5}{n(n+2)(n+4)(n+6)(n+8)} \end{pmatrix}.$$

$t$  has to be rational and positive. For every pair  $(n, s)$  from Table 2 we get a solution of the system and a polynomial equation of degree 5 whose positive

rational roots are the possible values for  $t$ . The only dimension in which we get a positive rational value for  $t$  is  $n = 56$  with  $t = 6$ . But then  $\Lambda^*$  would have to be even and hence  $\Lambda$  would be unimodular.  $\square$

**Lemma 14** *Let  $\Lambda$  be unimodular with  $\min(\Lambda) = 8$  and  $S(\Lambda)$  a spherical 11-design, then  $n = 72$  and  $\Lambda$  is even and extremal.*

*Proof*  $\Lambda$  is even (see Lemma 9). As the theta-series of even unimodular lattices are modular forms,  $n$  has to be divisible by eight and  $\min(\Lambda) \leq 2\lfloor \frac{n}{24} \rfloor + 2$  (Koecher and Krieg 1998, V.2.8.Satz). Therefore  $n = 72$  for  $m = 8$  is the only possible combination.  $\square$

## 5 13-design lattices of minimum $\leq 11$

We will now prove that there is no integral lattice with minimum smaller or equal to 11 whose minimal vectors form a 13-design. For minima smaller than 10 we can use the results for 11-designs.

**Lemma 15** *There is no integral lattice  $\Lambda$  with  $\min(\Lambda) < 10$  such that  $S(\Lambda)$  is a spherical 13-design.*

*Proof* If  $S(\Lambda)$  forms a 13-design it also forms an 11-design and hence can only be an extremal even unimodular lattice of dimension 24, 48 or 72. But as a result of (Martinet 2001, Proposition 4.1) we know that these lattices yield no higher designs.  $\square$

So the only statement left to prove is the following:

**Lemma 16** *There is no integral 13-design lattice of minimum 10 or 11.*

*Proof* If we assume that  $\Lambda$  would be an integral 13-design lattice with  $\min(\Lambda) \in \{10, 11\}$  then the following system of equations would have integral non-negative solutions for  $s$  and  $s_1, \dots, s_5$ .

$$\begin{pmatrix} 1 & 2^2 & 3^2 & 4^2 & 5^2 \\ 1 & 2^4 & 3^4 & 4^4 & 5^4 \\ 1 & 2^6 & 3^6 & 4^6 & 5^6 \\ 1 & 2^8 & 3^8 & 4^8 & 5^8 \\ 1 & 2^{10} & 3^{10} & 4^{10} & 5^{10} \\ 1 & 2^{12} & 3^{12} & 4^{12} & 5^{12} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix} = \begin{pmatrix} \frac{sm^2}{n} - m^2 \\ \frac{3sm^4}{n(n+2)} - m^4 \\ \frac{15sm^6}{n(n+2)(n+4)} - m^6 \\ \frac{105sm^8}{n(n+2)(n+4)(n+6)} - m^8 \\ \frac{945sm^{10}}{n(n+2)(n+4)(n+6)(n+8)} - m^{10} \\ \frac{10395sm^{12}}{n(n+2)(n+4)(n+6)(n+8)(n+10)} - m^{12} \end{pmatrix}.$$

But an easy calculation shows that there are no such solutions.  $\square$

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