Extremal Lattices and Hilbert Modular Forms

David Dursthoff, March 2017

Extremal lattices are remarkable objects of number theory. They define many of the densest known sphere packings, good spherical designs, and often have an interesting automorphism group. Important examples are the $\mathbb{E}_8$-lattice and the Leech lattice (both extremal unimodular), as well al the Barnes Wall lattice BW_{16} (extremal 2-modular) and the Coxeter Todd lattice (extremal 3-modular).

A fascinating property of even unimodular (or p-modular) lattice is that their theta series is an elliptic modular form under the whole modular group (or over the Fricke group of degree p). Since the theory of elliptic modular forms is well developed, this leads to many properties of lattices. So one can define extremal lattices as lattices whose theta series take the value 1 at infinity with the highest order.

Over totally real number fields, one can develop the same theory. We just consider the case of real quadratic number fields. We look at lattices (Λ, Q) over a number field which are distinguished into three types (depending on whether narrow and class number are the same, and whether the lattice is unimodular or trace unimodular). To a totally positive field element α we define a positive definite $\mathbb{Z}$-lattice $(\Lambda, \mathrm{tr}(\alpha Q))$, called trace lattices. To our lattice we associate two trace lattices (Λ_1, Q_1) and (Λ_2, Q_2), depending on the type. These two lattices describe (Λ, Q) uniquely.

We can thus define the theta series of (Λ, Q) as the merged theta series of the two trace lattices. This is first of all a formal power series in two variables q_1 and q_2. By suitable interpretation, this is a Hilbert modular form of level one.

In the same way a Hilbert modular form f can be written in a q-expansion, i.e. as a formal power series in q_1 and q_2. This give rise to computational implementations. Also we can order the coefficients using the lexicographic ordering. Since the space of Hilbert modular forms of given weight is finite dimensional, on can identify a unique extremal modular form, which again has the value 1 at infinity with the highest order.

A lattice is called extremal if its theta series is an extremal Hilbert modular form. In this thesis I construct, discuss, and classify extremal lattices over the number fields $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$.

Also I extend a method by Bachoc and Venkov. This method is based on calculations of so-called configuration numbers, which give the number of lattice vectors of the same length and scalar product with a fixed lattice vector. It does not use the concrete lattice, but one can extract equations of the configuration numbers from spherical theta series. They are Hilbert modular forms, hence the known ring structure of the modular forms yields equations of the configuration numbers. Often it is possible to compute all extremal lattices with given configuration numbers.

Numbers of known extremal lattices over real quadratic number fields.

Field	$\mathbb{Q}[\sqrt{5}]$	$\mathbb{Q}[\sqrt{2}]$	$\mathbb{Q}[\sqrt{3}]$	
Type	(i)	(i)	(ii)	(iii)
Dim. 2	-	-	1	-
4	1	1	2	1
6	-	-	1	-
8	2	1	3	≥ 1
10	-	-	21	-
12	1	5		≥ 1
16	≥ 2	≥ 1	0	0
20		≥ 1	0	0
24	≥ 1	≥ 1	0	0