Algorithms for the computation of Sato's b-functions in algebraic D-module theory

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Diplomarbeit

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“He had discovered a great law of human action, without knowing it – namely, that in order to make a man or a boy covet a thing, it is only necessary to make the thing difficult to obtain.”

Mark Twain (1835 – 1910),
“The Adventures of Tom Sawyer”, Chapter 2

Introduction

In the early 1970s, M. Sato introduced *a-, b- and c-functions* associated to *prehomogeneous vector spaces* [SS72]. Simultaneously and independently, J. Bernstein defined b-functions as part of the construction of a *meromorphic extension* of a certain real valued analytic function and proved that every polynomial has a non-zero b-function [Ber71, Ber72]. This b-function is known as the *Bernstein-Sato polynomial* today.
B. Malgrange pointed out a strong relation between the Bernstein-Sato polynomial and the *local monodromy* of a hypersurface given by a polynomial, if the hypersurface has only isolated singularities. In this case, all eigenvalues of the local monodromy at the origin are of the form $e^{-2\pi i\alpha}$, where α is a root of the Bernstein-Sato polynomial [Mal74, Mal75].
In 1976, M. Kashiwara showed that all roots of the Bernstein-Sato polynomial are negative rational numbers [Kas76]. Many special cases have been studied until T. Oaku gave a first algorithm to compute the Bernstein-Sato polynomial of an arbitrary polynomial in 1997 [Oak97c, Oak97a, Oak97b].
The focus of this work lies on algorithmical and computational aspects. One of the goals of this work is to give a clearly formulated and easy to understand introduction to the theory of b-functions.
We loosely follow the book by M. Saito, B. Sturmfels and N. Takayama [SST00] and use techniques and methods proposed by M. Noro [Nor02]. We have implemented the main algorithms in the computer algebra system SINGULAR [DGPS10], respectively its subsystem SINGULAR:PLURAL [GLS10] designed for computations in non-commutative polynomial algebras. The implementations are available in either one of the libraries `bfun.lib` [AL10], `dmodapp.lib` [LA10] or `dmod.lib` [LMM10]. These libraries are freely distributed together with SINGULAR. All examples presented in this work were computed using our implementations.
This work is structured as follows. We start in Chapter 1 by revisiting the theory of non-commutative Gröbner bases in G-algebras, studying fundamental properties of the Weyl algebra and giving an algebraic definition of the terms *b-function* and *Bernstein-Sato polynomial*. We will see that the computation of b-functions naturally splits up into two steps: computing the so-called *initial ideal* and intersecting it with a certain subalgebra. Chapter 2 deals with initial ideal. Moreover, the notion of the *Gel'fand-Kirillov dimension* is introduced. In addition, Chapter 3 is dedicated to the intersecting problem, though in a somewhat broader framework. In Chapter 4, we investigate Bernstein-Sato polynomials and prove Bernstein's Theorem. We also examine the other parts of what we call *Bernstein's data*. Some of the many applications of b-functions are addressed in Chapter 5. Finally, in Chapter 6 we describe the main procedures of our implementation

and compare it with the existing ones in the computer algebra systems Asir [NST06] and the D-module package [TL06] of Macaulay 2 [GS05]. Moreover, we perform experiments concerning certain approaches, we develop throughout this work.

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1 Basics

In this chapter, we introduce basic definitions and notations. Then we briefly revisit the theory of non-commutative *Gröbner bases* in *G-algebras* and study the most important properties of the *Weyl algebra*. Eventually, we define *b-functions* and *Bernstein-Sato polynomials*, which form the main point of interest of this work.

1.1 General notations

We use the notation $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for the natural numbers including zero and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ for the integers. The symbols $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ stand for the fields of the rational, real and complex numbers, respectively. By $\mathbb{K}$, we always mean an arbitrary field of characteristic zero.
For $v \in \mathbb{K}^n$ for $n \in \mathbb{N}$, we denote the i-th component of v by v_i, $1 \leq i \leq n$. We further set $vw := \sum_{i=1}^{n} v_i w_i$ as the standard scalar product of $v, w \in \mathbb{R}^n$ and $|v| := \sum_{i=1}^{n} v_i$ as the length of v.
Given a ring R, which is not necessarily commutative, and a subset $F \subseteq R$, we use the notation $\langle F \rangle := {}_R\langle F \rangle := R \cdot F$ for the left ideal , $\langle F \rangle_R := F \cdot R$ for the right ideal and ${}_R\langle F \rangle_R := R \cdot F \cdot R$ for the two-sided ideal in R generated by F.
By "ideal" and "module", we mean left ideal and left module, respectively, unless stated otherwise.

1.2 G-algebras and Gröbner bases

Definition 1.1. Let A be a $\mathbb{K}$-vector space with an additional binary operation $\cdot : A \times A \to A$. One calls A a $\mathbb{K}$-*algebra* if the following conditions hold for all $a, b, c \in A$ and for all $k, l \in \mathbb{K}$:

(a) There exists an element $1 \in A$ such that $1 \cdot a = a = a \cdot 1$,

(b) $(a + b) \cdot c = a \cdot c + b \cdot c$,

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$,

(d) $(ka) \cdot (lb) = (kl)(a \cdot b)$.

One calls A *associative*, if additionally

(e) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

and *commutative*, if

(f) $a \cdot b = b \cdot a$.

Definition 1.2. Let A, B be $\mathbb{K}$-algebras. A homomorphism of vector spaces $\phi : A \to B$, which also satisfies $\phi(1) = 1$ and $\phi(a \cdot a') = \phi(a) \cdot \phi(a')$ for all $a, a' \in A$ is called a *homomorphism* of $\mathbb{K}$-algebras.

Lemma 1.3. Let A, B be $\mathbb{K}$-algebras and $\phi : A \to B$ a homomorphism of $\mathbb{K}$-algebras. Then the *kernel* of ϕ, $\ker(\phi) := \{a \in A \mid \phi(a) = 0\}$, is a two-sided ideal of A.

Proof. We note that $\ker(\phi)$ is not empty since $\phi(0) = \phi(0 + 0) = \phi(0) + \phi(0)$, hence $0 \in \ker(\phi)$. Further, $\ker(\phi)$ is closed under addition since $\phi(a + a') = \phi(a) + \phi(a') = 0$ for $a, a' \in \ker(\phi)$. If $a \in \ker(\phi)$ and $r \in A$, then $\phi(r \cdot a) = \phi(r) \cdot \phi(a) = 0 = \phi(a) \cdot \phi(r) = \phi(a \cdot r)$. Hence, $r \cdot a, a \cdot r \in \ker(\phi)$. $\square$

Example 1.4. Consider n indeterminates $x_1, \ldots, x_n$ and the set of *monomials* (or *words*)

$$\mathcal{M} := \{x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_m}^{\alpha_m} \mid 1 \le i_1, \ldots, i_m \le n, m, \alpha_i \in \mathbb{N}_0\}.$$

Then the set

$$\mathcal{F}_n := \mathbb{K}\langle x_1, \ldots, x_n \rangle := \{\sum_{i=1}^{m} a_i m_i \mid a_i \in \mathbb{K}, m_i \in \mathcal{M}, m \in \mathbb{N}\}$$

consisting of all finite $\mathbb{K}$-linear combinations of monomials is an associative noncommutative $\mathbb{K}$-algebra with respect to the multiplication defined as concatenation, i. e.

$$x_{i_1}^{\alpha_1} \ldots x_{i_m}^{\alpha_m} \cdot x_{j_1}^{\beta_1} \ldots x_{j_l}^{\beta_l} := x_{i_1}^{\alpha_1} \ldots x_{i_m}^{\alpha_m} x_{j_1}^{\beta_1} \ldots x_{j_l}^{\beta_l}$$

for $x_{i_1}^{\alpha_1} \ldots x_{i_m}^{\alpha_m}, x_{j_1}^{\beta_1} \ldots x_{j_l}^{\beta_l} \in \mathcal{M}$.
One calls $\mathcal{F}_n$ the *free associative* $\mathbb{K}$-*algebra* and its elements *polynomials*. We write $\mathrm{Mon}(\mathcal{F}_n)$ instead of $\mathcal{M}$ for the set of monomials of $\mathcal{F}_n$.

Recall that a (strict partial) ordering on a set M is an irreflexive and transitive (and therefore asymmetric) relation on M.

Definition 1.5. Let M be a set.

(a) An ordering $\prec$ on M is called a *total ordering*, if either $m \prec m'$ or $m' \prec m$ holds for all $m, m' \in M, m \neq m'$.

(b) A total ordering $\prec$ on M is called a *well ordering* if every non-empty subset of M has a least element with respect to $\prec$.

(c) A total ordering $\prec$ on $\mathrm{Mon}(\mathcal{F}_n)$ is called a *monomial ordering* if it is compatible with the multiplication in the following sense: For all $f, g \in \mathrm{Mon}(\mathcal{F}_n)$ it holds that

  (i) $f \prec g$ implies $p \cdot f \cdot p' \prec p \cdot g \cdot p'$ for all $p, p' \in \mathrm{Mon}(\mathcal{F}_n)$.

  (ii) If $f = p \cdot g \cdot p'$ and $f \neq g$, then $g \prec f$.

In this situation, any $0 \neq f \in \mathcal{F}_n$ can be uniquely written as $f = c \cdot m + f'$ such that $0 \neq c \in \mathbb{K}$ and $m' \prec m$ for any monomial m' occurring in f with non-zero coefficient. We then call $\mathrm{lm}(f) := m$ the *leading monomial* of f.

Example 1.6. The standard ordering $<$ on $\mathbb{R}$ is a total ordering. But it is not a well ordering, since for instance $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ or $\mathbb{R}$ itself each have no least element. If we lift the standard ordering componentwise to $\mathbb{R}^n$, i. e. we define an ordering $<_{\mathrm{cw}}$ by $v <_{\mathrm{cw}} w$ if $v_i < w_i$ for all $1 \leq i \leq n$, then $<_{\mathrm{cw}}$ is not even a total ordering, since for instance in the case $n = 2$, $(1, 0)$ and $(0, 1)$ are incomparable.

Note that for a two-sided ideal $T \subseteq \mathcal{F}_n$ the *quotient* $\mathcal{F}_n/T$ is itself a well-defined $\mathbb{K}$-algebra.

Definition 1.7. Let $T \subseteq \mathcal{F}_n$ be a two-sided ideal, generated by elements of the form

$$x_j x_i - c_{ij} x_i x_j - d_{ij},\ 1 \leq i < j \leq n,$$

where $0 \neq c_{ij} \in \mathbb{K}$ and $d_{ij} \in \mathcal{F}_n$ is a polynomial involving only *standard monomials*, i. e. monomials of the form $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m}$. The factor algebra

$$A := \mathcal{F}_n/T =: \mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij} \mid 1 \leq i < j \leq n\}\rangle$$

is called a *G-algebra* if the following two conditions hold:

(a) *Ordering condition:* There exists a monomial ordering $\prec$ on $\mathrm{Mon}(\mathcal{F}_n)$ such that $\mathrm{lm}(d_{ij}) \prec x_i x_j$ for all $1 \leq i < j \leq n$.

(b) *Non-degeneracy condition:* For all $1 \leq i < j < k \leq n$ it holds that

$$c_{ik} c_{jk} \cdot d_{ij} x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk} = 0.$$

A G-algebra is said to be of *Lie type* if $c_{ij} = 1$ for all $1 \leq i < j \leq n$.

By convention, in the notation $\mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij} \mid 1 \leq i < j \leq n\}\rangle$ we will only mention the non-commutative relations and omit the commutative ones.

Example 1.8.

(a) By setting $c_{ij} := 1$ and $d_{ij} := 0$ for all $1 \leq i < j \leq n$, we obtain the commutative polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$.

(b) The G-algebra D_n in $2n$ indeterminates defined by $c_{ij} := 1$ and $d_{ij} := \delta_{i+n,j} = \begin{cases} 1 & j = i+n \\ 0 & j \neq i+n \end{cases}$ for all $1 \leq i < j \leq 2n$ is called the *n-th Weyl algebra* over $\mathbb{K}$.

We will take a closer look at the Weyl algebra in the next section. For now, we focus on G-algebras in general.

Theorem 1.9 ([Lev05, Theorem 1.4.7.]). Let A be a G-algebra.

(a) A is left and right Noetherian.

(b) A is an integral domain.

(c) A has both left and right quotient rings.

It follows from (a) that every left (or right or two-sided, respectively) ideal in a G-algebra is finitely generated.

Theorem 1.10 ([Lev05, Lemma 1.2.2.]). Let A be a G-algebra in n indeterminates $x_1, \dots, x_n$. Then A has a *Poincaré-Birkhoff-Witt basis* (or *PBW basis* for short), i. e. A is generated as a $\mathbb{K}$-vector space by the set $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_1, \dots \alpha_n \in \mathbb{N}_0\}$ of *standard monomials*.

Throughout this work, we will make frequent use of *multi-index notations*, meaning we will simply write x^α for the standard monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

Definition 1.11. As an important consequence of Theorem 1.10, we obtain the result that any non-zero element f of a G-algebra A can be uniquely written in terms of standard monomials:

$$f = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha, \quad c_\alpha \in \mathbb{K}.$$

We then call

$$\deg(f) := \max_{\alpha \in \mathbb{N}_0^n} \{|\alpha| \mid c_\alpha \neq 0\}$$

the *(total) degree* of f and more general, for a given $0 \neq w \in \mathbb{R}^n$, we call

$$\deg_w(f) := \max_{\alpha \in \mathbb{N}_0^n} \{\sum_{i=1}^{n} w_i \alpha_i \mid c_\alpha \neq 0\}$$

the *weighted (total) degree* of f. As a convention, we put $\deg_w(0) := \deg(0) := -\infty$.

Definition 1.12. Let A be a G-algebra.

(a) Adopting Definition 1.5(c), a total ordering $\prec$ on the standard monomials of A is called a *monomial ordering* if $x^\alpha \prec x^\beta$ implies $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$ for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$.

(b) A *global ordering* is a monomial ordering $\prec$ satisfying $1 \prec x^\alpha$ for all $0 \neq \alpha \in \mathbb{N}_0^n$.

(c) We say that x^α *divides* x^β, if $\alpha \leq_{\mathrm{cw}} \beta$ and denote it by $x^\alpha \mid x^\beta$.

Example 1.13. The two following examples are both global orderings.

- The *lexicographical ordering* defined by $x^\alpha \prec_{\text{lex}} x^\beta$ if there exists $1 \leq i \leq n$ such that $\alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i$.
- The *degree reverse lexicographical ordering* defined by $x^\alpha \prec_{\text{degrevlex}} x^\beta$ if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and there exists $1 \leq i \leq n$ such that $\alpha_n = \beta_n, \ldots, \alpha_{i+1} = \beta_{i+1}, \alpha_i > \beta_i$.

Definition 1.14. Let A be a G-algebra.

(a) If $\prec$ is a monomial ordering, any $0 \neq f \in A$ can be uniquely written as $f = c \cdot x^\alpha + f'$ such that $0 \neq c \in \mathbb{K}$ and $x^{\alpha'} \prec x^\alpha$ for any non-zero term $c' \cdot x^{\alpha'}$ of f'. We then call

$$\begin{aligned} \operatorname{lm}(f) &:= x^\alpha && \text{the } \textit{leading monomial} \text{ of } f, \\ \operatorname{lc}(f) &:= c && \text{the } \textit{leading coefficient} \text{ of } f, \\ \operatorname{le}(f) &:= \alpha && \text{the } \textit{leading exponent} \text{ of } f \\ \text{and } \operatorname{lt}(f) &:= c \cdot x^\alpha && \text{the } \textit{leading term} \text{ of } f. \end{aligned}$$

(b) For any subset $F \subseteq A$, the $\mathbb{K}$-vector space

$$L(F) := \mathbb{K} \cdot \{\operatorname{lm}(f) \mid f \in F\} \subseteq A$$

is called the *span of leading monomials* of F.

(c) An element $f \in A$ is called *reduced* with respect to a subset $S \subseteq A$, if no monomial in f is contained in $L(S)$.

If we have to deal with more than one ordering at the same time, we write $\operatorname{lm}_\prec(f)$ instead of $\operatorname{lm}(f)$ etc. in order to avoid confusion.

Lemma 1.15. A monomial ordering on a G-algebra A is a well ordering if and only if it is a global ordering.

Proof. Let $\prec$ be a monomial well ordering on A and let x^λ be the least standard monomial with respect to $\prec$. Suppose $x^\lambda \prec 1 = x^0$. Then $x^{(k+1)\lambda} \prec x^{k\lambda}$ for all $k \in \mathbb{N}$, since $\prec$ is a monomial order. But $\ldots \prec x^{(k+1)\lambda} \prec x^{k\lambda} \prec \ldots x^\lambda \prec x^0 = 1$ is an infinite descending sequence, which contradicts that $\prec$ is a well ordering.
Conversely, let $\prec$ be a global ordering on A and let $\emptyset \neq S \subseteq A$. Consider $\mathcal{L}(S) := \{\operatorname{le}(s) \mid s \in S\} \subseteq \mathbb{N}_0^n$. By Dickson's Lemma (e. g. [GP08]) there exists a finite subset $M \subseteq \mathcal{L}(S)$ such that for all $\beta \in \mathcal{L}(S)$ there is an element $\alpha \in M$ with $\alpha \leq_{\text{cw}} \beta$. Without loss of generality, we may assume $M = \{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}\}$ such that $x^{\alpha^{(1)}} \prec x^{\alpha^{(2)}} \prec \ldots \prec x^{\alpha^{(m)}}$. Let $s \in S$ with $\operatorname{lm}(s) = x^\alpha$. Then there exists some i, $1 \leq i \leq m$, such that $\alpha^{(i)} \leq_{\text{cw}} \alpha$, i. e. there exists $\gamma \in \mathbb{N}_0^n$ such that $\alpha^{(i)} + \gamma = \alpha$. Thus, $1 = x^0 \prec x^\gamma$ and $\prec$ is a monomial ordering yield $x^{\alpha^{(1)}} \preceq x^{\alpha^{(i)}} \preceq x^{\alpha^{(i)}+\gamma} = x^\alpha$. Hence, we have $p \preceq s$ for any $p \in S$ with $le(p) = \alpha^{(1)}$. Proceeding the same way with $S' := \{p - \operatorname{lt}(p) \in S \mid le(p) = \alpha^{(1)}\}$, we inductively find the least element of S, since any polynomial in A has only finitely many terms. $\square$

Definition 1.16. Let $\prec$ be a global ordering on a G-algebra A. Further let $I \subseteq A$ be a left ideal and $\emptyset \neq G \subseteq I$ a finite subset. G is called a *(left) Gröbner basis* of I with respect to $\prec$ if for every $0 \neq f \in I$ there exists $g \in G$ such that $\operatorname{lm}(g) \mid \operatorname{lm}(f)$.
A Gröbner basis G is called *reduced*, if for all $g, g' \in G$, $g \neq g'$, the following conditions hold:

- $\operatorname{lc}(g) = 1$,
- $0 \notin G$ and $\operatorname{lm}(g) \nmid \operatorname{lm}(g')$,
- $\operatorname{lm}(g) \nmid m$ for every monomial m in $g' - \operatorname{lt}(g')$.

We emphasize that we explicitly require well orderings in the definition of Gröbner bases.

Theorem 1.17. Let A be a G-algebra, $\prec$ a global ordering on A and $I \subseteq A$ an ideal. Then there exists a Gröbner basis of I with respect to $\prec$.

We refer to [Lev05] for a proof and algorithms.

Definition 1.18. Let $\mathcal{G}$ be the set of all non-empty finite ordered subsets of a G-algebra A with respect to a global ordering $\prec$. A (left) *normal form* on A is a map

$$\mathrm{NF} : A \times \mathcal{G} \to A, \quad (f, G) \mapsto \mathrm{NF}(f, G)$$

satisfying the following conditions for all $f \in A, G \in \mathcal{G}$:

(a) $\mathrm{NF}(0, G) = 0$.

(b) $\mathrm{NF}(f, G) \neq 0$ implies $\operatorname{lm}(\mathrm{NF}(f, G)) \notin L(G)$.

(c) $f - \mathrm{NF}(f, G) \in {}_A\langle G\rangle$.

A normal form NF is called *reduced* with respect to $G \in \mathcal{G}$ if $\mathrm{NF}(f, G)$ is reduced with respect to G in the sense of Definition 1.14(c) for all $f \in A$.

Lemma 1.19. Let A be a G-algebra, $I \subseteq A$ an ideal and $G \subseteq I$ a Gröbner basis of I.

(a) For $f \in A$, we have $f \in I$ if and only if $\mathrm{NF}(f, G) = 0$.

(b) If $\mathrm{NF}(\cdot, G)$ is reduced, then it is unique.

(c) If $\mathrm{NF}(\cdot, G)$ is reduced, then it is $\mathbb{K}$-linear.

Proof.

(a) If $\mathrm{NF}(f, G) = 0$, 1.18(c) yields $f \in \langle G\rangle = I$. If $\mathrm{NF}(f, G) \neq 0$, then $\operatorname{lm}(\mathrm{NF}(f, G)) \notin L(G) = L(I)$ by 1.18(b), and hence $\mathrm{NF}(f, G) \notin I$. Thus, $f \notin I$ by 1.18(c).

(b) Let $f \in A$ and let r, r' be two reduced normal forms of f with respect to G. Using 1.18(c), it holds that $(f-r)-(f-r') = r'-r \in I$. Suppose that $r \neq r'$. But then $\operatorname{lm}(r'-r) \in L(I)$. Since $\operatorname{lm}(r'-r)$ is a monomial of either r or r', this normal form is not reduced, which contradicts the assumption.

(c) Let $f, g \in A$, $k \in \mathbb{K}$. By 1.18(c), we have that

$$kf + g - NF(kf+g, G) \in I \text{ and}$$
$$k(f - NF(f,G)) + g - NF(g,G) = kf + g - kNF(f,G) - NF(g,G) \in I.$$

But then also $p := NF(kf+g, G) - (kNF(k,G) + NF(g,G)) \in I$, which is possible if and only if $p = 0$ because $\operatorname{NF}(\cdot, G)$ is reduced by assumption. □

Definition 1.20. Let A be a G-algebra, $0 \neq f, g \in A$ with $\operatorname{lm}(f) = x^\alpha$ and $\operatorname{lm}(g) = x^\beta$. Further, consider $\gamma \in \mathbb{N}_0^n$ defined by $\gamma_i := \max\{\alpha_i, \beta_i\}$ for all $1 \leq i \leq n$. We then call

$$\operatorname{spoly}(f,g) := x^{\gamma-\alpha} f - \frac{\operatorname{lc}(x^{\gamma-\alpha} f)}{\operatorname{lc}(x^{\gamma-\beta} g)} x^{\gamma-\beta} g$$

the (left) *s-polynomial* of f and g.

The following theorem gives a collection of characterizations of Gröbner bases. We refer again to [Lev05] for proofs.

Theorem 1.21 (Characterization of Gröbner bases). Let A be a G-algebra, $I \subseteq A$ an ideal and $G = \{g_1, \ldots, g_s\} \subseteq I$ a set. Then the following statements are equivalent:

(a) G is a Gröbner basis of I.

(b) $\operatorname{NF}(f, G) = 0$ for all $f \in I$.

(c) Each $f \in I$ has a *standard representation* with respect to G, i. e. there exist $a_1, \ldots, a_s \in A$ such that f can be written as

$$f = \sum_{i=1}^{s} a_i g_i,$$

where $\operatorname{lm}(a_i g_i) \prec f$ and $a_i g_i \neq 0$ for all $1 \leq i \leq s$.

(d) *Buchberger's Criterion* holds: $\operatorname{NF}(\operatorname{spoly}(g_i, g_j), G) = 0$ for all $1 \leq i, j \leq s$.

(e) The equality $L(I) = L(G)$ holds.

Dealing with non-commutative algebras, the notion of the Lie bracket can be quite useful.

Lemma 1.22. Let A be a G-algebra. For two elements $f, g \in A$, we denote the *Lie bracket* of f, g by $[f, g] := fg - gf$. The Lie bracket is bilinear, alternating and fulfills the *Jacobi identity*, i. e. $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$ for all $f, g, h \in A$.

The following Generalized Product Criterion is useful both in theory and practice.

Lemma 1.23 (Generalized Product Criterion [LS03]). Let A be a G-algebra of Lie type and $f, g \in A$. If $\mathrm{lm}(f)$ and $\mathrm{lm}(g)$ have no common factors, then $\mathrm{spoly}(f, g)$ reduces to $[f, g]$ with respect to the set $\{f, g\}$.

1.3 The Weyl Algebra

Recall from Example 1.8(b) that the n-th Weyl algebra over $\mathbb{K}$ is defined to be the G-algebra

$$D_n := \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_i x_j = x_j \partial_i + \delta_{ij} \mid 1 \leq i, j \leq n\}\rangle.$$

As mentioned above, we will make use of multi-index notation. In case of the n-the Weyl algebra, we will abbreviate $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ by x^α and analogously $\partial_1^{\beta_1} \ldots \partial_n^{\beta_n}$ by ∂^β. In the case $n = 1$, we will simply write x and ∂ instead of x_1 and ∂_1.

Lemma 1.24. In the first Weyl algebra we have for all $i, j \in \mathbb{N}$

$$\partial^i x^j = \sum_{k=0}^{\min(i,j)} \frac{i! \cdot j!}{k! \cdot (i-k)! \cdot (j-k)!} x^{j-k} \partial^{i-k}. \tag{1.1}$$

Proof. We prove the lemma by induction on i and j.
First, let $i = 1$. Then (1.1) can be written as

$$\partial x^j = x^j \partial + j x^{j-1}. \tag{1.2}$$

For $j = 1$, there is nothing to show. Assume (1.2) holds for j. Using $\partial x = x\partial + 1$, we obtain

$$\begin{aligned} \partial x^{j+1} &= (\partial x^j) x = (x^j \partial + j x^{j-1}) x = x^j \partial x + j x^j = x^j (x\partial + 1) + j x^j \\ &= x^{j+1} \partial + (j+1) x^j. \end{aligned}$$

Hence, (1.2) holds by induction.

Now, assume (1.1) holds for i. We have

$$\begin{aligned}
\partial^{i+1}x^j &= \partial(\partial^i x^j) \\
&= \sum_{k=0}^{\min(i,j)} \frac{i!\cdot j!}{k!\cdot(i-k)!\cdot(j-k)!}\partial x^{j-k}\partial^{i-k} \\
&= \sum_{k=0}^{\min(i,j)} \frac{i!\cdot j!}{k!\cdot(i-k)!\cdot(j-k)!}\left(x^{j-k}\partial+(j-k)x^{j-k-1}\right)\partial^{i-k} \\
&= \sum_{k=0}^{\min(i,j)} \frac{i!\cdot j!}{k!\cdot(i-k)!\cdot(j-k)!}x^{j-k}\partial^{i+1-k} \\
&\quad + \sum_{k=0}^{\min(i,j)} \frac{i!\cdot j!}{k!\cdot(i-k)!\cdot(j-k)!}(j-k)x^{j-k-1}\partial^{i-k}
\end{aligned}$$

Considering the second sum, we note that the last summand equals zero if and only if $\min(i,j)=j$. So we may express this by using the *Kronecker symbol* $\delta_{j,k}$.

$$\begin{aligned}
&\sum_{k=0}^{\min(i,j)} \frac{i!\cdot j!}{k!\cdot(i-k)!\cdot(j-k)!}(j-k)x^{j-k-1}\partial^{i-k} \\
&= \sum_{k=0}^{\min(i,j)} \frac{\delta_{j,k}\cdot i!\cdot j!}{k!\cdot(i-k)!\cdot(j-k-1)!}x^{j-k-1}\partial^{i-k} \\
&= \sum_{k=1}^{\min(i,j)+1} \frac{\delta_{j,k-1}\cdot i!\cdot j!}{(k-1)!\cdot(i+1-k)!\cdot(j-k)!}x^{j-k}\partial^{i+1-k}.
\end{aligned}$$

Considering both sums again, we get

$$\begin{aligned}
\partial^{i+1}x^j &= x^j\partial^{i+1} + \frac{\delta_{j,\min(i,j)}\cdot i!\cdot j!}{(\min(i,j))!\cdot(i-\min(i,j))!\cdot(j-\min(i,j)-1)!}x^{j-\min(i,j)-1}\partial^{i-\min(i,j)} \\
&\quad + \sum_{k=1}^{\min(i,j)}\left(\frac{i!\cdot j!}{k!\cdot(i-k)!\cdot(j-k)!}+\frac{i!\cdot j!}{(k-1)!\cdot(i+1-k)!\cdot(j-k)!}\right)x^{j-k}\partial^{i+1-k} \\
&= x^j\partial^{i+1} + \frac{\delta_{j,\min(i,j)}\cdot j!}{(j-i-1)!}x^{j-i-1} + \sum_{k=1}^{\min(i,j)} \frac{i!\cdot j!\cdot(i+1-k)+i!\cdot j!\cdot k}{k!\cdot(i+1-k)!\cdot(j-k)!}x^{j-k}\partial^{i+1-k} \\
&= x^j\partial^{i+1} + \frac{\delta_{j,\min(i,j)}\cdot j!}{(j-i-1)!}x^{j-i-1} + \sum_{k=1}^{\min(i,j)} \frac{(i+1)!\cdot j!}{k!\cdot(i+1-k)!\cdot(j-k)!}x^{j-k}\partial^{i+1-k} \\
&= \sum_{k=0}^{\min(i+1,j)} \frac{(i+1)!\cdot j!}{k!\cdot(i+1-k)!\cdot(j-k)!}x^{j-k}\partial^{i+1-k},
\end{aligned}$$

which concludes the proof. $\square$

The lemma gives rise to a nice and useful identity.

Corollary 1.25. In the n-th Weyl algebra we have the following identity for $p \in \mathbb{K}[x_1, \dots, x_n]$:

$$[\partial_i, p] = \frac{\partial p}{\partial x_i},$$

where $\frac{\partial p}{\partial x_i}$ stands for the formal partial derivative of p with respect to x_i.

Proof. Let $p = \sum_{j=0}^{m} c_j x_i^j$, where $c_j \in \mathbb{K}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$. Then we obtain by using equation (1.2)

$$\partial_i p = \sum_{j=0}^{m} c_j \partial_i x_i^j = \sum_{j=0}^{m} c_j (x_i^j \partial_i + j x_i^{j-1}) = \sum_{j=0}^{m} c_j x_i^j \partial_i + \sum_{j=0}^{m} j c_j x_i^{j-1} = p\partial_i + \frac{\partial p}{\partial x_i},$$

which was to be shown. □

Corollary 1.26. In the first Weyl algebra, let $\theta := x\partial$ be the *Euler operator*. Then we have for all $m \in \mathbb{N}$ the identity

$$x^m \partial^m = \prod_{i=0}^{m-1} (\theta - i).$$

Proof. For $m = 1$, there is nothing to show. Suppose the claim holds for $m \in \mathbb{N}$. Then Equation (1.2) implies

$$\begin{aligned} x^{m+1}\partial^{m+1} &= x x^m \partial \partial^m = x(\partial x^m - m x^{m-1})\partial^m = x\partial x^m \partial^m - m x^m \partial^m \\ &= (\theta - m) \prod_{i=0}^{m-1} (\theta - i) = \prod_{i=0}^{m} (\theta - i). \end{aligned}$$

Hence the claim follows by induction. □

Corollary 1.27. In the first Weyl algebra, $x^j \partial^i$ can be written as $p \cdot x$ for some $p \in D_1$, if $j > i$.

Proof. Let $\theta := x\partial$. By Equation (1.2),

$$\begin{aligned} x^m(\theta - k) &= x^{m+1}\partial - k x^m = \partial x^{m+1} - (m+1)x^m - k x^m \\ &= (\partial x - m - 1 - k)x^m = (\theta - m - k)x^m \end{aligned}$$

for all $m, k \in \mathbb{N}$. Since $j - i > 0$, applying Corollary 1.26 yields

$$x^j \partial^i = x^{j-i} x^i \partial^i = x^{j-i} \prod_{k=0}^{i-1} (\theta - k) = \left(\prod_{k=0}^{i-1} (\theta - j + i - k) \right) x^{j-i}.$$ □

Remark 1.28. By Theorem 1.10 any $p \in D_n$ can be written in the form

$$p = \sum_{\alpha,\beta \in \mathbb{N}_0^n} c_{\alpha\beta} x^\alpha \partial^\beta,$$

where $c_{\alpha\beta} \in \mathbb{K}$ such that $c_{\alpha\beta} \neq 0$ for only finitely many pairs (α, β). Therefore, there exists a natural isomorphism ψ' of $\mathbb{K}$-vector spaces between the commutative polynomial ring in $2n$ indeterminates, $\mathbb{K}[x, \partial]$, and the Weyl algebra, given by

$$\psi' : \mathbb{K}[x, \partial] \to D_n, \quad x^\alpha \partial^\beta \mapsto x^\alpha \partial^\beta.$$

In order to avoid irritating notations when dealing with partial derivatives with respect to ∂_i, we relabel ∂_i to ξ_i when necessary. Formally, instead of ψ' in the Remark above, we consider $\psi : \mathbb{K}[x, \xi] \to D_n, x^\alpha \xi^\beta \mapsto x^\alpha \partial^\beta$.

Using this isomorphism, we can formulate the Leibniz Rule for efficient computation in D_n as a generalization of Lemma 1.24.

Theorem 1.29 (Leibniz Rule).
Let ψ be the isomorphism from the previous remark. For $f, g \in \mathbb{K}[x, \xi]$, we have

$$\psi(f) \cdot \psi(g) = \sum_{k \in \mathbb{N}_0^n} \frac{1}{k_1! \cdots k_n!} \cdot \psi\left(\frac{\partial^k f}{\partial \xi^k} \cdot \frac{\partial^k g}{\partial x^k}\right).$$

Proof. Both sides of the equation are $\mathbb{K}$-bilinear. Hence it suffices to prove the claim for monomials, say $f = x^\alpha \xi^\beta, g = x^\gamma \xi^\delta$. Then we have

$$\psi(f) \cdot \psi(g) = x^\alpha (\partial^\beta \cdot x^\gamma) \partial^\delta.$$

On the other hand,

$$\begin{aligned}
&\sum_{k \in \mathbb{N}_0^n} \frac{1}{k_1! \cdots k_n!} \cdot \psi\left(\frac{\partial^k (x^\alpha \xi^\beta)}{\partial \xi^k} \cdot \frac{\partial^k (x^\gamma \xi^\delta)}{\partial x^k}\right) \\
&= x^\alpha \cdot \left(\sum_{k \in \mathbb{N}_0^n} \frac{1}{k_1! \cdots k_n!} \cdot \psi\left(\frac{\partial^k (\xi^\beta)}{\partial \xi^k} \cdot \frac{\partial^k (x^\gamma)}{\partial x^k}\right)\right) \cdot \partial^\delta.
\end{aligned}$$

Hence we can assume that $f = \xi^\beta$ and $g = x^\gamma$. Further,

$$\psi(\xi^\beta) \cdot \psi(x^\gamma) = \psi(\xi_1^{\beta_1} \cdots \xi_n^{\beta_n}) \cdot \psi(x_1^{\gamma_1} \cdots x_n^{\gamma_n}) = \prod_{i=1}^n \psi(\xi_i^{\beta_i}) \cdot \psi(x_i^{\gamma_i})$$

and

$$\sum_{k \in \mathbb{N}_0^n} \frac{1}{k_1! \cdots k_n!} \cdot \psi\left(\frac{\partial^k f}{\partial \xi^k} \cdot \frac{\partial^k g}{\partial x^k}\right) = \prod_{i=1}^n \sum_{k_i \in \mathbb{N}_0} \frac{1}{k_i!} \cdot \psi\left(\frac{\partial^{k_i} \xi_i^{\beta_i}}{\partial \xi_i^{k_i}} \cdot \frac{\partial^{k_i} x_i^{\gamma_i}}{\partial x_i^{k_i}}\right).$$

Thus, it suffices to prove the case $n = 1$, i. e.

$$\partial^i x^j = \sum_{k\in\mathbb{N}_0} \frac{1}{k!} \cdot \psi\left(\frac{\partial^k \xi^i}{\partial \xi^k} \cdot \frac{\partial^k x^j}{\partial x^k}\right).$$

Using the well known identity for the k-th derivative of a monomial in one indeterminate, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} \frac{1}{k!} \cdot \psi\left(\frac{\partial^k \xi^i}{\partial \xi^k} \cdot \frac{\partial^k x^j}{\partial x^k}\right) &= \sum_{k=0}^{\min(i,j)} \frac{1}{k!} \cdot \psi\left(\frac{i!}{(i-k)!}\xi^{i-k} \cdot \frac{j!}{(j-k)!}x^{j-k}\right)\\ &= \sum_{k=0}^{\min(i,j)} \frac{i! \cdot j!}{k! \cdot (i-k)! \cdot (j-k)!} x^{j-k}\partial^{i-k},\end{aligned}$$

The claim then follows by Lemma 1.24. □

As an immediate consequence of the Leibniz Rule we get the following result about the total degree.

Corollary 1.30. For all $p, q \in D_n$, we have $\deg(p \cdot q) = \deg(p) + \deg(q)$. Moreover, the Weyl algebra is a domain.

Proof. Examining the Leibniz Rule, the summand of maximal total degree is the one for $k = 0$, namely $\psi(\psi^{-1}(p)\psi^{-1}(q))$. Since the degree is invariant under ψ, we have reduced the claim to the commutative case, where its correctness is known. The second claim follows from the first one using the degree argument. □

Corollary 1.31. The group of units of D_n equals $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.

Proof. Every $k \in \mathbb{K}^*$ is a unit. Let $p, q \in D_n$ such that $pq = 1$. Then $0 = \deg(p) + \deg(q)$ by Corollary 1.30. Since $\deg(p), \deg(q) \geq 0$, we have $p, q \in \mathbb{K}^*$. □

Theorem 1.32. The Weyl algebra is *simple*, i. e. $\{0\}$ and D_n itself are the only two-sided ideals in D_n.

Proof. Let $I \subseteq D_n$ be a non-zero two-sided ideal. Then there exists an element $0 \neq p \in I$ such that $k := \deg(\mathrm{lm}(p)) \leq \deg(\mathrm{lm}(q))$ for all $q \in I$ for some fixed ordering. If $k = 0$, then $p \in \mathbb{K}$ and thus $I = D_n$. So, assume $k > 0$. Let $\mathrm{lm}(p) = x^\alpha \partial^\beta$. If $\alpha_i \neq 0$ for some $1 \leq i \leq n$, then Corollary 1.25 yields $0 \neq \frac{\partial p}{\partial x_i} = [\partial_i, p] \in I$ and $\deg(\mathrm{lm}([\partial_i, p])) < k$, which contradicts the minimality of k. Hence $\alpha = 0$. But then $\beta_i \neq 0$ for some $1 \leq i \leq n$. In this case x_i and p do not commute, i. e. $0 \neq [x_i, p] \in I$ and $\deg([x_i, p]) < k$, again contradicting the minimality of k to conclude the proof. □

Corollary 1.33. Let A be a non-zero $\mathbb{K}$-algebra and $\phi : D_n \to A$ a homomorphism of $\mathbb{K}$-algebras. Then ϕ is injective.

Proof. Since kernels of homomorphisms of $\mathbb{K}$-algebras are two-sided ideals by Lemma 1.3, and D_n is simple, it follows that $\ker(\phi) = \{0\}$. □

Remark 1.34. The famous *Dixmier conjecture* [Dix68] states that every endomorphism of the Weyl algebra is surjective, and hence because of the previous corollary, an automorphism.

1.4 Global b-functions

Definition 1.35. Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$. For a non-zero polynomial

$$p = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta \in D_n$$

we call

$$\mathrm{in}_{(-w,w)}(p) := \sum_{\alpha,\beta:\ -w\alpha + w\beta = \deg_{(-w,w)}(p)} c_{\alpha\beta} x^\alpha \partial^\beta.$$

the *initial form* of p with respect to $(-w, w)$. We further set $\mathrm{in}_{(-w,w)}(0) := 0$.
This definition extends in a natural way to a set of polynomials $F \subseteq D_n$:

$$\mathrm{in}_{(-w,w)}(F) := \{\mathrm{in}_{(-w,w)}(p) \mid p \in F\}$$

is called the *initial form* of F with respect to $(-w, w)$.

Definition 1.36. Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$ and $I \subseteq D_n$ be an ideal. For $s := \sum_{i=1}^n w_i x_i \partial_i$ we consider the intersection $\mathrm{in}_{(-w,w)}(I) \cap \mathbb{K}[s]$, which is an ideal in the principal ideal domain $\mathbb{K}[s]$. Its monic generator $b_{I,w}(s)$ is called the *global b-function* of I with respect to the weight w.

In order to see, that $\mathrm{in}_{(-w,w)}(I) \cap \mathbb{K}[s]$ is indeed an ideal, we refer to the next chapter. It is known that the global b-function of an important class of ideals (so called *holonomic* ones) is non-zero – independent of the choice for the weight vector. We will prove this in Chapter 3.

Example 1.37. Consider the second Weyl algebra $D_2 = \mathbb{K}\langle x, y, \partial_x, \partial_y \mid \{\partial_y y = y\partial_y + 1, \partial_x x = x\partial_x + 1\}\rangle$ and the ideal $I = \langle 3x^2\partial_y + 2y\partial_x, 2x\partial_x + 3y\partial_y + 6\rangle \subseteq D_2$. This is the annihilator of $\frac{1}{x^3-y^2}$, where x, y act via multiplication and ∂_x and ∂_y act via partial derivation with respect to x and y, respectively (see also Chapter 4). We compute the global b-functions of I with respect to the weights $(1, 0), (0, 1)$ and $(2, 3)$ using Singular. Each output is a list consisting of the roots of the global b-function and their corresponding multiplicities.

```
LIB "bfun.lib";                            // load the library
ring r = 0,(x,y,Dx,Dy),dp;                 // create commutative ring
def D2 = Weyl(); setring D2;               // create Weyl algebra
ideal I = 3*x^2*Dy+2*y*Dx, 2*x*Dx+3*y*Dy+6; // define ideal
intvec w = 1,0; bfctIdeal(I,w);            // define w and compute b_{I,w}
==>[1]:
==>   _[1]=0
==>   _[2]=-3/2
==>[2]:
==>   1,1
w = 0,1; bfctIdeal(I,w);
==>[1]:
==>   _[1]=0
==>   _[2]=-4/3
==>   _[3]=-2/3
==>[2]:
==>   1,1,1
w = 2,3; bfctIdeal(I,w);
==>[1]:
==>   _[1]=-6
==>[2]:
==>   1
```

Hence,

$$b_{I,(1,0)}(s) = s(s+\frac{3}{2}), \quad b_{I,(0,1)}(s) = s(s+\frac{4}{3})(s+\frac{2}{3}), \quad b_{I,(2,3)}(s) = s+6.$$

One way to define a b-function for a polynomial is to apply the global b-function for a specific ideal and a specific weight vector.

Definition 1.38. For a polynomial $f \in \mathbb{K}[x_1,\dots,x_n]$ the ideal

$$I_f := \langle t-f, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t, \dots, \partial_n + \frac{\partial f}{\partial x_n}\partial_t \rangle \subseteq D_n\langle t, \partial_t \rangle$$

in the $(n+1)$-th Weyl-Algebra $D_n\langle t,\partial_t\rangle$ is called the *Malgrange ideal* of f.

Definition 1.39. Let $f \in \mathbb{K}[x_1,\dots,x_n]$, I_f be the Malgrange ideal of f and $w = (1,0,\dots,0) \in \mathbb{R}^{n+1}$ such that the weight of ∂_t is 1. Then

$$b_f(s) := (-1)^{\deg(b_{I_f,w})} b_{I_f,w}(-s-1)$$

is called the *global b-function* or the *Bernstein-Sato polynomial* of f.

As of yet, it seems that the substitution of s by $-s-1$ in the definition of the Bernstein-Sato polynomial does not make much sense. We will see the reason for it in Chapter 4.

Example 1.40. Let us calculate the Bernstein-Sato polynomial of the zero polynomial. The Malgrange ideal $I_0 = \langle t, \partial\rangle \subsetneq D_1\langle t, \partial_t\rangle$ equals $\mathrm{in}_{(-w,w)}(I_0)$ for $w = (1,0)$ and $\partial_t \cdot t = t\partial_t + 1 \in I_0$. Hence, the b-function of I_0 with respect to w equals $s+1$ and thus, $b_0(s) = s$.
Now, let us determine the Bernstein-Sato polynomial for a constant $0 \neq c \in \mathbb{K}$. Since $t - c \in I_f$, we have $-c = \mathrm{in}_{(-w,w)}(t-c) \in \mathrm{in}_{(-w,w)}(I_c)$ for $w = (1,0)$ and we conclude that $b_c(s) = 1$.

Example 1.41. Consider $f := x^3 - y^2 \in \mathbb{K}[x, y]$. We compute the Bernstein-Sato polynomial of f.

```
LIB "bfun.lib";
ring r = 0,(x,y),dp;
poly f = x^3-y^2;
bfct(f);
==>[1]:
==>   _[1]=-1
==>   _[2]=-5/6
==>   _[3]=-7/6
==>[2]:
==>   1,1,1
```

Hence, $b_f(s) = (s+1)(s+\frac{5}{6})(s+\frac{7}{6})$.

The goal of the following chapters is to carefully explain how the procedures `bfctIdeal` and `bfct` used in Examples 1.37 and 1.41, respectively, work. Since the Bernstein-Sato polynomial is a special case of the global b-function of an ideal, we will deal with the latter first.
Following its definition, the computation of the global b-function of an ideal $I \subseteq D_n$ with respect to a weight w can be tackled in two steps:

(a) Compute $J := \mathrm{in}_{(-w,w)}(I)$.

(b) Compute the intersection of J with the subalgebra $\mathbb{K}[s]$.

We will discuss both steps separately, starting with the computation of $\mathrm{in}_{(-w,w)}(I)$ in Chapter 2, while Chapter 3 is dedicated to the intersection problem. Then Chapter 4 entirely concerns Bernstein-Sato polynomials before we eventually turn our attention to some of the interesting applications of b-functions in Chapter 5.

2 Initial ideals

In this chapter, we define what an *initial ideal* is and we investigate how to compute it. We do this in a somewhat bigger framework than needed to compute the global b-function of an ideal. However, there are interesting applications related to the global b-function which require this more general setting.

2.1 Filtrations and gradings

Definition 2.1. A *filtered ring* is a ring R together with a family $F := \{F_i \mid i \in \mathbb{Z}\}$ of subgroups of the additive group of R such that the following conditions hold:

(a) $F_i \cdot F_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$,

(b) $F_i \subseteq F_j$ for all $i < j \in \mathbb{Z}$, and

(c) $R = \bigcup_{i \in \mathbb{Z}} F_i$.

The family F is called a *filtration* of R.

Example 2.2. Consider the finite dimensional vector space of elements in $\mathbb{K}[x_1, \ldots, x_n]$ of total degree at most k, $F_k := \{p \in \mathbb{K}[x_1, \ldots, x_n] \mid \deg(p) \leq k\}$. Note that $F_k = \{0\}$ for all $k < 0$ since $\deg(0) = -\infty$ by convention and 0 is the only polynomial of negative degree. Then $F := \{F_k \mid k \in \mathbb{Z}\}$ is a filtration on $\mathbb{K}[x_1, \ldots, x_n]$, the *degree filtration*. Similarly, the degree filtration on D_n, obtained by substituting $\mathbb{K}[x_1, \ldots, x_n]$ with D_n in the definition above, is also called the *Bernstein filtration*.

Definition 2.3. We call a non-zero vector $(u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ a *weight vector* for the Weyl algebra D_n if $u_i + v_i \geq 0$ for all $1 \leq i \leq n$.

Throughout this work, we will suppose that u_i is the weight for the generator x_i and v_i is the weight for the generator ∂_i. We will see the reason for not permitting weight vectors with $u_i + v_i < 0$ for $1 \leq i \leq n$ in Remark 2.9 below.
An important generalization of the Bernstein filtration is the V-filtration.

Example 2.4. Let $(u, v) \in \mathbb{R}^{2n}$ be a weight vector and for all $m \in \mathbb{Z}$ consider the set $V_m := \{p \in D_n \mid \deg_{(u,v)}(p) \leq m\}$ of all elements of D_n whose total weighted degree does not exceed m. Then $V := \{V_m \mid m \in \mathbb{Z}\}$ is a filtration on D_n, the *V-filtration* with respect to (u, v).

Definition 2.5. A *graded ring* is a ring R together with a family $G = \{G_i \mid i \in \mathbb{Z}\}$ of subgroups of the additive group of R such that

(a) $G_i \cdot G_j \subseteq G_{i+j}$ for all $i, j \in \mathbb{Z}$, and

(b) $R = \bigoplus\limits_{i \in \mathbb{Z}} G_i$.

The family G is called a *grading* of R.
In this situation, any $r \in R$ can be uniquely written in the form $r = \sum_{i \in \mathbb{Z}} g_i$ for some $g_i \in G_i$. The element g_i is called the i-th *homogeneous component* of r. Moreover, r is said to be *homogeneous* if r consists of only one homogeneous component, i. e. there exists some $i \in \mathbb{Z}$ such that $r \in G_i$.

Remark 2.6. Any graded ring $R = \bigoplus_{i \in \mathbb{Z}} G_i$ has a natural filtration $F = \{F_i \mid i \in \mathbb{Z}\}$, where $F_i = \bigoplus_{j \leq i} G_j$.
Conversely, let R_F be a filtered ring with filtration $F = \{F_i \mid i \in \mathbb{Z}\}$. We construct a graded ring $\mathrm{gr}(R_F)$ as follows. Set

$$G_m := F_m / F_{m-1}, \quad m \in \mathbb{Z}, \qquad \text{and} \qquad \mathrm{gr}(R_F) := \bigoplus_{m \in \mathbb{Z}} G_m.$$

To define a multiplication in $\mathrm{gr}(R_F)$, it suffices to consider homogeneous elements. If $r \in F_m$ and $r \notin F_{m-1}$, then r is said to have degree m and $[r] = r + F_{m-1} \in G_m$ is the leading term of r. Suppose s has degree m'. We set $[r] \cdot [s] := rs + F_{m+m'-1} \in G_{m+m'}$. This multiplication is well-defined since

$$[r] \cdot [s] = \begin{cases} [rs] & \text{if } rs \notin F_{m+m'-1} \\ [0] & \text{otherwise.} \end{cases}$$

The ring $\mathrm{gr}(R_F)$ obtained with respect to this multiplication is called the *associated graded ring* of R_F.

Example 2.7. Consider the V-filtration from Example 2.4 and let $\mathrm{gr}_{(u,v)}(D_n)$ denote the associated graded ring of D_n with respect to this filtration. It holds that

$$x_i \in V_{u_i} \setminus V_{u_i - 1}, \qquad \partial_i \in V_{v_i} \setminus V_{v_i - 1} \qquad \text{and} \qquad 1 \in V_0 \setminus V_{-1}.$$

We have $[x_i][\partial_i] = x_i \partial_i + V_{u_i + v_i - 1}$ and $[\partial_i][x_i] = \partial_i x_i + V_{u_i + v_i - 1} = x_i \partial_i + 1 + V_{u_i + v_i - 1}$.
Assume, $u_i + v_i > 0$. Then $u_i + v_i - 1 > -1$ and thus $[\partial_i][x_i] = x_i \partial_i + V_{u_i + v_i - 1}$. Hence $[x_i]$ and $[\partial_i]$ commute in $\mathrm{gr}_{(u,v)}(D_n)$.
Now let $u_i + v_i = 0$. Then $[\partial_i][x_i] = x_i \partial_i + 1 + V_{-1}$. Hence $[x_i]$ and $[\partial_i]$ do not commute. This implies that

$$\mathrm{gr}_{(u,v)}(D_n) \cong \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_i x_j = x_j \partial_i + \delta_{i,j} \delta_{u_i + v_i, 0}\}\rangle.$$

In particular,

$$\begin{aligned} \mathrm{gr}_{(u,v)}(D_n) &\cong \mathbb{K}[x,\partial] && \text{if} \quad 0 <_{\mathrm{cw}} u+v \text{ and} \\ \mathrm{gr}_{(u,v)}(D_n) &\cong D_n && \text{if} \quad 0 = u+v, \text{ i. e. } u=-v. \end{aligned}$$

Remark 2.8. Let R be a graded ring. A (left) ideal I of R is called a *graded (left) ideal* if all the homogeneous components of each element of I also belong to I, i. e. I can be generated by homogeneous elements.
Now suppose that R is a filtered ring with filtration $F = \{F_i \mid i \in \mathbb{Z}\}$. Associated with any left ideal I of R, there is a *graded left ideal* $\mathrm{gr}(I)$ of $\mathrm{gr}(R)$ which is defined by setting

$$(\mathrm{gr}(I))_n := (I + F_{n-1}) \cap F_n / F_{n-1} \subseteq F_n / F_{n-1} \qquad \text{and} \qquad \mathrm{gr}(I) := \bigoplus_n (\mathrm{gr}(I))_n.$$

Note that $(I+F_{n-1}) \cap F_n / F_{n-1} \cong I \cap F_n / I \cap F_{n-1}$. If $[a] \in (\mathrm{gr}(I))_n$ and $[r] \in (\mathrm{gr}(R))_m$ then $[r][a] = ra + F_{m+n-1} \in (I + F_{m+n-1}) \cap F_{m+n}/F_{m+n-1}$. This shows that $\mathrm{gr}(I)$ is indeed a left ideal of $\mathrm{gr}(R)$.

Remark 2.9. Recall that the Weyl algebra is defined as the free associative algebra $\mathbb{K}\langle x, \partial\rangle$ modulo the two-sided ideal of relations $T = \langle \partial_i x_j - x_j \partial_i - \delta_{ij} \mid 1 \leq i,j \leq n\rangle$ (Example 1.8(b)). The V-filtration with respect to (u,v) on D_n as defined in Example 2.4 is induced by a corresponding filtration on the free associative algebra. Assume we have a weight vector (u,v) with $u_i + v_i < 0$ for some $1 \leq i \leq n$. Then $\partial_i x_i - x_i \partial_i - 1 \in \mathbb{K}\langle x, \partial\rangle$ is inhomogeneous of degree 0 with highest homogeneous component -1. Thus, we have $1 \in \mathrm{gr}_{(u,v)}(T)$ and therefore, $\mathrm{gr}_{(u,v)}(D_n) = \mathrm{gr}_{(u,v)}(\mathbb{K}\langle x,\partial\rangle)/\mathrm{gr}_{(u,v)}(T) = \{0\}$. Hence, these weights are not interesting for us.

Recall that we have introduced the notion of initial forms with respect to a weight vector $(-w, w)$ in Definition 1.35. The following definition generalizes this concept to arbitrary weight vectors for the Weyl algebra.

Definition 2.10. Consider again the V-filtration with respect to the weight vector (u,v). For a non-zero polynomial

$$p = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta \in D_n$$

we call

$$\mathrm{in}_{(u,v)}(p) := \sum_{\alpha,\beta:\ u\alpha + v\beta = \deg_{(u,v)}(p)} c_{\alpha\beta} x^\alpha \partial^\beta \in \mathrm{gr}_{(u,v)}(D_n)$$

the *initial form* of p with respect to (u,v). For the zero polynomial, we set $\mathrm{in}_{(u,v)}(0) := 0$. For an ideal $I \subseteq D_n$, we call the $\mathbb{K}$-vector space generated by all initial forms of elements of I with respect to (u,v),

$$\mathrm{in}_{(u,v)}(I) := \mathbb{K} \cdot \{\mathrm{in}_{(u,v)}(p) \mid p \in I\} \subseteq \mathrm{gr}_{(u,v)}(D_n),$$

the *initial ideal* of I with respect to (u,v).

We need to show that the initial ideal carries its name for a reason.

Lemma 2.11. If I is an ideal in D_n, then $\mathrm{in}_{(u,v)}(I)$ is an ideal in $\mathrm{gr}_{(u,v)}(D_n)$ and $\mathrm{gr}_{(u,v)}(I) = \mathrm{in}_{(u,v)}(I)$ holds.

Proof. The latter claim follows from the first one by definition.
We set $x^\alpha_{[i,j]} := x_i^{\alpha_i} \cdots x_j^{\alpha_j}$ for $1 \leq i \leq j \leq n$. Without loss of generality let $u_i + v_i = 0$ for $1 \leq i \leq m$ and $u_j + v_j > 0$ for $m+1 \leq j \leq n$. Further, let $p \in I$ and $r \in D_n$ be (u,v)-homogeneous. Then we can identify r with its canonical projection $\mathrm{in}_{(u,v)}(r)$ onto $\mathrm{gr}_{(u,v)}(D_n)$. We consider monomials $x^\alpha \partial^\beta$ and $x^\gamma \partial^\delta$ of r and $\mathrm{in}_{(u,v)}(p)$, respectively. By Example 2.7, x_i and ∂_i commute in $\mathrm{gr}_{(u,v)}(D_n)$ for $m+1 \leq i \leq n$. Thus, in $\mathrm{gr}_{(u,v)}(D_n)$ it holds that

$$x^\alpha \partial^\beta \cdot x^\gamma \partial^\delta = x^{\alpha+\gamma}_{[m+1,n]} \partial^{\beta+\delta}_{[m+1,n]} \cdot x^\alpha_{[1,m]} \partial^\beta_{[1,m]} \cdot x^\gamma_{[1,m]} \partial^\delta_{[1,m]}.$$

Applying the Leibniz Rule (Theorem 1.29) shows that any monomial in $x^\alpha_{[1,m]} \partial^\beta_{[1,m]} \cdot x^\gamma_{[1,m]} \partial^\delta_{[1,m]}$ has the form

$$\psi \left(\frac{\partial^k x^\alpha \xi^\beta}{\partial \xi^k}_{[1,m]} \cdot \frac{\partial^k x^\gamma \xi^\delta}{\partial x^k}_{[1,m]} \right) = x^\alpha_{[1,m]} \cdot \psi \left(\frac{\partial^k \xi^\beta}{\partial \xi^k}_{[1,m]} \cdot \frac{\partial^k x^\gamma}{\partial x^k}_{[1,m]} \right) \cdot \partial^\delta_{[1,m]}.$$

Let us analogously denote $w_{[i,j]} := (w_i, \ldots, w_j), 1 \leq i \leq j \leq n$, for $w \in \mathbb{R}^n$. It follows that the weighted total degree with respect to (u,v) of each term in $x^\alpha_{[1,m]} \partial^\beta_{[1,m]} \cdot x^\gamma_{[1,m]} \partial^\delta_{[1,m]}$ equals

$$\begin{aligned} &u_{[1,m]}((\alpha+\gamma)_{[1,m]} - k) + v_{[1,m]}((\delta+\beta)_{[1,m]} - k) \\ =&u_{[1,m]}(\alpha+\gamma)_{[1,m]} + v_{[1,m]}(\delta+\beta)_{[1,m]} - k(u+v)_{[1,m]} \\ =&u_{[1,m]}(\alpha+\gamma)_{[1,m]} + v_{[1,m]}(\delta+\beta)_{[1,m]}. \end{aligned}$$

Since we also have

$$\deg_{(u,v)}(x^{\alpha+\gamma}_{[m+1,n]} \partial^{\beta+\delta}_{[m+1,n]}) = u_{[m+1,n]}(\alpha+\gamma)_{[m+1,n]} + v_{[m+1,n]}(\delta+\beta)_{[m+1,n]}$$

it follows that

$$\deg_{(u,v)}(x^\alpha \partial^\beta \cdot x^\gamma \partial^\delta) = u(\alpha+\gamma) + v(\delta+\beta).$$

Thus, $r \cdot \mathrm{in}_{(u,v)}(p)$ is (u,v)-homogeneous of degree $\deg_{(u,v)}(r) + \deg_{(u,v)}(p)$ and $r \cdot \mathrm{in}_{(u,v)}(p) = \mathrm{in}_{(u,v)}(r \cdot p) \in \mathrm{in}_{(u,v)}(I)$. $\square$

Note that we have already encountered an initial ideal in the definition of the global b-function (Definition 1.36), namely in the special case where the weight vector (u,v) is of the form $(-w,w)$. In this case the associated graded ring is isomorphic to the Weyl algebra according to Example 2.7, allowing us to identify D_n and $\mathrm{gr}_{(-w,w)}(D_n)$, which subsequently justifies that we have not mentioned graded rings earlier. We will keep the practice of identifying D_n and $\mathrm{gr}_{(-w,w)}(D_n)$ below. Moreover, now it is clear, that the intersection $\mathrm{in}_{(-w,w)}(I) \cap \mathbb{K}[s]$ in Definition 1.36 is indeed an ideal in $\mathbb{K}[s]$.

2.2 Gel'fand-Kirillov dimension

A crucial tool in the study of D-modules is the *Gel'fand-Kirillov dimension.* In this section, we define it and give a couple of its properties, which will be vital later on. Let us start by revisiting filtrations.

Definition 2.12. Let R be a filtered ring with filtration $F = \{F_i \mid i \in \mathbb{Z}\}$. A *filtration* of a (left) R-module M is a family $\{M_i \mid i \in \mathbb{Z}\}$ of subgroups of M satisfying

(a) $M_i \subseteq M_j$ for $i < j$,

(b) $F_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}$, and

(c) $\bigcup_{i \in \mathbb{Z}} M_i = M$.

A module with a filtration is called a *filtered module*.

Remark 2.13. Given a filtered ring R and a filtered R-module M, analogous definitions as in Remark 2.6 result in a *graded* $\mathrm{gr}(R)$*-module* $\mathrm{gr}(M)$ *associated* to M. In particular, $\mathrm{gr}(M) = \bigoplus_{i \in \mathbb{Z}} (\mathrm{gr}(M))_i$ where $(\mathrm{gr}(M))_i = M_i / M_{i-1}$.

Definition 2.14. Let A be a $\mathbb{K}$-algebra with a filtration $A' = \{A_i \mid i \in \mathbb{Z}\}$ and M a (left) A-module with a filtration $M' = \{M_i \mid i \in \mathbb{Z}\}$.
One calls A' *standard* if $A_i = A_1^i$ for all i and *finite dimensional* if $A_0 = \mathbb{K}$ and $\dim_{\mathbb{K}}(A_i) < \infty$ for all i.
One calls M' *standard* if $M_i = A_i M_0$ for all i and *finite dimensional* if $\dim_{\mathbb{K}}(M_i) < \infty$ for all i.

Example 2.15.

(a) The degree filtration on $\mathbb{K}[x_1, \ldots, x_n]$ and the Bernstein filtration on D_n both are standard and finite dimensional.

Consider $V^i := \{p \in \mathbb{K}[x_1, \ldots, x_n] \mid \deg(p) = i\}$ and put $V := V^1$. Clearly, $F := \{F_i \mid i \in \mathbb{Z}\}$ is a standard filtration, where $F_i := \{0\}$ for $i < 0$, $F_0 := V^0 = \mathbb{K}$ and $F_i := \bigoplus_{k=0}^{i} V^k$ for $i > 0$. Further, $\{x_1, \ldots, x_n\}$ is a $\mathbb{K}$-basis of V and more general $\{x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \mid i_k \in \mathbb{N}_0, \sum_{k=1}^{n} i_k = i\}$ is a $\mathbb{K}$-basis of V^i. Hence, $\dim_{\mathbb{K}}(V^i)$ equals the number of *partitions* of i, which equals the number of *multisets* of cardinality i with elements taken from a set of cardinality n, but this is known by basic combinatorics to be exactly $\binom{i+n-1}{n-1}$. Similarly, $\dim_{\mathbb{K}}(F_i) = \binom{i+n}{n}$.

(b) The V-filtration on D_n is neither standard nor finite dimensional in general. Depending on the chosen weight vector (u, v), $u_1 = 0$ is possible for instance. In this case, we have $\deg_{(u,v)}(x_1) = 0$ and hence $x_1^k \in A_0$ for all $k \in \mathbb{N}$. Thus, $\mathbb{K} \neq A_0$ and also $\dim_{\mathbb{K}}(A_0) = \infty$.

Remark 2.16. Let A be a finitely generated $\mathbb{K}$-algebra. Then there exists a $\mathbb{K}$-subspace $V \subseteq A$ such that A is generated by V as $\mathbb{K}$-algebra. Such a V induces a standard finite dimensional filtration $\{A_i \mid i \in \mathbb{Z}\}$ on A by setting $A_i := \{0\}$ for $i < 0$, $A_0 := V^0 := \mathbb{K}$ and $A_i := \sum_{j=0}^{i} V^j$.
If M is a finitely generated (left) A-module, there exists a subspace M_0 such that $M = RM_0$. Then M has a standard finite dimensional filtration $\{M_i \mid i \in \mathbb{Z}\}$ with $M_i := \{0\}$ for $i < 0$ and $M_i := M_0 R_i$ for $i > 0$.

Definition 2.17. In the situation of Remark 2.16, we call

$$\operatorname{GKdim}(A) := \limsup_{i\to\infty} \log_i(\dim_{\mathbb{K}}(A_i)) = \limsup_{i\to\infty} \frac{\ln(\dim_{\mathbb{K}}(A_i))}{\ln(i)}$$

the *Gel'fand-Kirillov dimension* of A and

$$\operatorname{GKdim}(M) := \limsup_{i\to\infty} \log_i(\dim_{\mathbb{K}}(M_i)) = \limsup_{i\to\infty} \frac{\ln(\dim_{\mathbb{K}}(M_i))}{\ln(i)}$$

the *Gel'fand-Kirillov dimension* of M.

Before we see some examples for the Gel'fand-Kirillov dimension, we need to show that it is well defined, i. e. the Gel'fand-Kirillov dimension is invariant under the chosen standard finite dimensional filtration. To do so, we first need a lemma from analysis.

Lemma 2.18. Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be two sequences. If $g(n) \leq f(an+b)$ for some $a, b \in \mathbb{N}$ and sufficiently large n, then $\limsup_{n\to\infty} \log_n(g(n)) \leq \limsup_{n\to\infty} \log_n(f(n))$.

Proof. Let $\epsilon > 0$ and put $\lambda := \limsup_{n\to\infty} \log_n(f(n))$. Then

$$f(n) = n^{\log_n(f(n))} < n^{\epsilon + \limsup\limits_{n\to\infty} \log_n(f(n))} = n^{\epsilon+\lambda}$$

for sufficiently large n. This implies

$$g(n) \leq f(an+b) < (an+b)^{\epsilon+\lambda} = n^{\epsilon+\lambda}(a + \frac{b}{n})^{\epsilon+\lambda} < n^{2\epsilon+\lambda},$$

and thus, $\log_n g(n) < 2\epsilon + \lambda$, which yields the claim. $\square$

Theorem 2.19. The Gel'fand-Kirillov dimension does not depend on the chosen standard finite dimensional filtration.

Proof. Let $\tilde{V}, \tilde{M}_0$ be some other generating spaces for A and M and let $\tilde{A}', \tilde{M}'$ be the corresponding filtrations as above. There exists some $a \in \mathbb{N}$ such that $\tilde{V} \subseteq A_a$ as $\bigcup_{k\in\mathbb{N}_0} A_k = A$. Analogously, $\tilde{M}_0 \subseteq M_b$ for some $b \in \mathbb{N}$. Thus, $\tilde{M}_k = \tilde{M}_0\tilde{A}_k \subseteq M_b M_{ak} \subseteq M_{ak+b}$. Hence, $\dim_{\mathbb{K}}(\tilde{M}_k) \leq \dim_{\mathbb{K}}(M_{ak+b})$. On the other hand, switching roles yields $\dim_{\mathbb{K}}(\tilde{M}_k) \geq \dim_{\mathbb{K}}(M_{ak+b})$. The claim now follows from the previous lemma by setting $f(k) = \dim_{\mathbb{K}}(\tilde{M}_k)$ and $g(k) = \dim_{\mathbb{K}}(M_k)$. $\square$

Theorem 2.20. Let A be a G-algebra in n indeterminates. Then $\mathrm{GKdim}(A) = n$. Especially, $\mathrm{GKdim}(D_n) = 2n$ and $\mathrm{GKdim}(\mathbb{K}[x_1, \ldots, x_n]) = n$.

Proof. Consider $V^i := \{p \in A \mid \deg(p) = i\}$. Since A has a PBW-basis (Theorem 1.10), the same reasoning as in Example 2.15 shows that

$$\dim_{\mathbb{K}}(V^i) = \binom{i+n-1}{n-1} = \frac{(i+n)\cdot(i+n-1)\cdot\ldots\cdot(i+1)}{n!} =: p(i)$$

for some $p \in \mathbb{Q}[i]$. We have $\mathrm{lt}(p) = \frac{1}{n!}i^n$ and thus,

$$\mathrm{GKdim}(A) = \limsup_{i\to\infty} \frac{\ln(\dim_{\mathbb{K}}(V^i))}{\ln(i)} = \limsup_{i\to\infty} \frac{\ln(i^n)}{\ln(i)} = \limsup_{i\to\infty} \frac{n\ln(i)}{\ln(i)} = n. \qquad \square$$

We give a collection of properties of the Gel'fand-Kirillov dimension regarding related structures and refer to [MR01] for proofs.

Theorem 2.21. Let A be a G-algebra and M a (left) A-module.

(a) $\mathrm{GKdim}(M) \leq \mathrm{GKdim}(A)$.

(b) If A' is a G-subalgebra of A, then $\mathrm{GKdim}(A') \leq \mathrm{GKdim}(A)$.

(c) If M' is a submodule of M, then $\mathrm{GKdim}(M') \leq \mathrm{GKdim}(M)$.

(d) For any standard finite dimensional filtration on A and M, respectively,

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(\mathrm{gr}(A)) \quad \text{and} \quad \mathrm{GKdim}(M) = \mathrm{GKdim}(\mathrm{gr}(M)).$$

In particular, a (left) D_n-module has a Gel'fand-Kirillov dimension of at most $2n$. There is also a lower bound for its dimension, known as Bernstein's inequality.

Theorem 2.22 (Bernstein's inequality).
Let M be a (left) D_n-module. Then $\mathrm{GKdim}(M) \geq n$.

Proofs can be found for instance in [MR01, Proposition 8.5.5] and [Cou95, Theorem 9.4.2].
Modules of minimal dimension are of special importance.

Definition 2.23. A D_n-module of Gel'fand-Kirillov dimension n is called *holonomic*. A (left) ideal $I \subseteq D_n$ is said to be *holonomic* if D_n/I is a holonomic D_n-module.

For example, for $0 \neq f \in \mathbb{K}[x_1, \ldots, x_n]$, the Malgrange ideal I_f from Definition 1.38 is holonomic as we will see later (Theorem 4.7).
There are cases where the Gel'fand-Kirillov dimension of a module over a non-commutative ring can be reduced to a purely commutative problem.

Theorem 2.24. Let A be a G-algebra in n indeterminates $x_1, \ldots, x_n$, $I \subseteq A$ an ideal and $M := A/I$ viewed as an A-module. Consider the isomorphism of vector spaces

$$\psi : A \to \mathbb{K}[x_1, \ldots, x_n], x^\alpha \mapsto x^\alpha$$

(cf. Lemma 1.10, Remark 1.28). Then $\operatorname{GKdim}(M) = \dim(\mathbb{K}[x_1, \ldots, x_n]/\psi(L(I)))$, where $L(I)$ stands for the span of leading monomials of I (cf. Definition 1.14(b)) and dim denotes the *Krull dimension*, i. e. the number of inclusions in a maximal strict chain of prime ideals.

The details can be found in [BGTV03]. For the Krull dimension, see e. g. [GP08]. For our purposes, it suffices to know that the Krull dimension of $\mathbb{K}[x_1, \ldots, x_n]$ equals n.

2.3 Weighted homogenization

Recall from Definitions 1.35 and 2.10 respectively, that for a weight vector $0 \neq (u, v) \in \mathbb{R}^{2n}$ satisfying $0 \leq_{\text{cw}} u + v$ and an ideal $I \subseteq D_n$ the initial ideal $\operatorname{in}_{(u,v)}(I) \subseteq \operatorname{gr}_{(u,v)}(D_n)$ arises by removing all terms which are not of maximal total weighted degree with respect to (u, v) for each element of I.
In order to compute the initial ideal, the method of weighted homogenization has been proposed by Noro [Nor02], which we will describe below. Homogenization in the Weyl algebra has also been studied by Castro-Jiménez and Narváez-Macarro in a more general context [CJNM97]. Let us start by deforming the Weyl algebra.

Definition 2.25. Let $\zeta, \eta \in \mathbb{R}^n_{>0}$. The G-algebra

$$D^{(h)}_{n,(\zeta,\eta)} := \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, h \mid \{\partial_j x_i = x_i \partial_j + \delta_{ij} h^{\zeta_i + \eta_j}\}\rangle$$

is called the n-th *weighted homogenized Weyl algebra* with respect to the *homogenization weights* ζ, η.

Note that x_i and ∂_i get weights ζ_i and η_i, respectively, and h gets weight 1.
As of yet, we have two different kinds of weight vectors, a weight vector $(u, v) \in \mathbb{R}^{2n}$ for the Weyl algebra coming from the initial ideal (or the V-filtration, respectively) and homogenization weights $\zeta, \eta \in \mathbb{R}^n_{>0}$ as introduced here.
Further note, that the relation $\partial_i x_i = x_i \partial_i + h^{\zeta_i + \eta_i}$ in $D^{(h)}_{n,(\zeta,\eta)}$ is homogeneous of degree $\zeta_i + \eta_i$, which is strictly positive, while the relation $\partial_i x_i = x_i \partial_i + 1$ in D_n is homogeneous of degree zero.
Having a homogenized algebra, we also need homogenized elements as well as homogenized orderings.

Definition 2.26. For $p = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta \in D_n$ one defines the *weighted homogenization* of p to be

$$H_{(\zeta,\eta)}(p) := \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta h^{\deg_{(\zeta,\eta)}(p) - (\zeta\alpha + \eta\beta)} \in D^{(h)}_{n,(\zeta,\eta)}.$$

This definition naturally extends to a set of polynomials.

As a convention, for $p \in D^{(h)}_{n,(\zeta,\eta)}$, we denote by $\deg_{(\zeta,\eta)}(p)$ the weighted total degree of p with respect to weights ζ, η for x, ∂ and weight 1 for h.

Definition 2.27. For a monomial ordering $\prec$ on D_n, we define an *associated homogenized ordering* $\prec^{(h)}$ on $D^{(h)}_{n,(\zeta,\eta)}$ by setting

$$h \prec^{(h)} x_i, \quad h \prec^{(h)} \partial_i, \quad 1 \leq i \leq n, \quad \text{and}$$

$$\begin{aligned} p \prec^{(h)} q \quad \text{if} \quad & \deg_{(\zeta,\eta)}(p) < \deg_{(\zeta,\eta)}(q) \\ \text{or} \quad & \deg_{(\zeta,\eta)}(p) = \deg_{(\zeta,\eta)}(q) \quad \text{and} \quad p_{|h=1} \prec q_{|h=1}. \end{aligned}$$

Lemma 2.28. The associated homogenized ordering is a well ordering.

Proof. The ordering $\prec^{(h)}$ is a monomial ordering because $\prec$ is one. It is global since (ζ, η) is strictly positive, which implies that $\deg_{(\zeta,\eta)}(1) = 0 < \deg_{(\zeta,\eta)}(x^\alpha \partial^\beta h^\lambda)$ for all $\alpha, \beta \in \mathbb{N}_0^n, \lambda \in \mathbb{N}_0$ not all simultaneously zero. The claim follows by Lemma 1.15. $\square$

Note that $\prec^{(h)}$ is a well ordering, regardless of whether $\prec$ is one. Further, for $\zeta = \eta = (1, \ldots, 1)$ the weighted homogenization corresponds to the *standard homogenization* as in [SST00]. The latter can be viewed as a natural generalization of homogenization in the commutative case as in [Laz83].
Analogue statements of the following two theorems can be found in [SST00] and [Nor02] respectively. Due to our different conception of Gröbner bases arising from the fact that we require well orderings, not just monomial ones, we provide new proofs for them.

Theorem 2.29. Let $F \subseteq D_n$ be a finite set and $\prec$ a global ordering. If $G^{(h)}$ is a Gröbner basis of $\langle H_{(\zeta,\eta)}(F) \rangle \subseteq D^{(h)}_{n,(\zeta,\eta)}$ with respect to $\prec^{(h)}$, then the *dehomogenization* of $G^{(h)}$, ${G^{(h)}}_{|h=1}$, is a Gröbner basis of $\langle F \rangle$ with respect to $\prec$.

Proof. By definition, for any $f \in \langle F \rangle$ with $\mathrm{lm}_{\prec^{(h)}}(H_{(\zeta,\eta)}(f)) = x^\alpha \partial^\beta h^\lambda$, there exists $g^{(h)} \in G^{(h)}$ with $\mathrm{lm}_{\prec^{(h)}}(g^{(h)}) = x^\gamma \partial^\delta h^\kappa$ satisfying $\mathrm{lm}_{\prec^{(h)}}(g^{(h)}) \mid \mathrm{lm}_{\prec^{(h)}}(H_{(\zeta,\eta)}(f))$. Since $H_{(\zeta,\eta)}(f)$ is (ζ, η)-homogeneous, $\deg_{(\zeta,\eta)}(m) = \zeta\alpha + \eta\beta + \lambda$ for all monomials m of $H_{(\zeta,\eta)}(f)$, which implies $m_{|h=1} \prec x^\alpha \partial^\beta$ according to the definition of $\prec^{(h)}$. Then

$$\mathrm{lm}(g^{(h)})_{|h=1} = x^\gamma \partial^\delta \mid x^\alpha \partial^\beta = \mathrm{lm}(H_{(\zeta,\eta)}(f))_{|h=1} = \mathrm{lm}(f),$$

which proves the claim. $\square$

Lemma 2.30. There is a bijection between the global orderings in D_n and $\mathrm{gr}_{(u,v)}(D_n)$.

Proof. Recall the examination of $\mathrm{gr}_{(u,v)}(D_n)$ in Example 2.7 and consider the isomorphism of $\mathbb{K}$-vector spaces (cf. Theorem 1.10, Lemma 1.28)

$$\begin{aligned} \psi : D_n &\to \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_i x_j = x_j \partial_i + \delta_{i,j} \delta_{u_i+v_i,0}\} \rangle \cong \mathrm{gr}_{(u,v)}(D_n), \\ x^\alpha \partial^\beta &\mapsto x^\alpha \partial^\beta. \end{aligned}$$

Let $\prec$ be a global ordering on D_n. We define an ordering $\prec_{\mathrm{gr}_{(u,v)}}$ on $\mathrm{gr}_{(u,v)}(D_n)$ by setting

$$x^\alpha\partial^\beta \prec_{\mathrm{gr}_{(u,v)}} x^\gamma\partial^\delta \quad \text{if} \quad \psi^{-1}(x^\alpha\partial^\beta) \prec \psi^{-1}(x^\gamma\partial^\delta).$$

Since ψ is an isomorphism and $\prec$ is global, $\prec_{\mathrm{gr}_{(u,v)}}$ is a well defined global ordering, too. Conversely, starting with a global ordering on $\mathrm{gr}_{(u,v)}(D_n)$, an analogue definition yields a global ordering on D_n. □

By the previous lemma, we are able to identify global orderings on D_n and $\mathrm{gr}_{(u,v)}(D_n)$. We come to the main result of this chapter, the interaction of the three $\mathbb{K}$-algebras D_n, $\mathrm{gr}_{(u,v)}(D_n)$ and $D^{(h)}_{n,(\zeta,\eta)}$.

Theorem 2.31. Let $I \subseteq D_n$ be an ideal, $\prec$ be a global ordering on $\mathrm{gr}_{(u,v)}(D_n)$ and $\prec_{(u,v)}$ the (not necessarily global) monomial ordering defined by

$$\begin{aligned} x^\alpha\partial^\beta \prec_{(u,v)} x^\gamma\partial^\delta \quad &\text{if} \quad u\alpha+v\beta < u\gamma+v\delta \\ &\text{or} \quad u\alpha+v\beta = u\gamma+v\delta \quad \text{and} \quad x^\alpha\partial^\beta \prec x^\gamma\partial^\delta. \end{aligned}$$

If $G^{(h)}$ is a Gröbner basis of $H_{(\zeta,\eta)}(I)$ with respect to $\prec^{(h)}_{(u,v)}$, then $\mathrm{in}_{(u,v)}(G^{(h)}|_{h=1})$ is a Gröbner basis of $\mathrm{in}_{(u,v)}(I)$ with respect to $\prec$.

Proof. The concepts of initial forms with respect to (u,v) and homogenized orderings are compatible in the following sense: Let $p \in D^{(h)}_{n,(\zeta,\eta)}$ be (ζ,η)-homogeneous. Then

$$\mathrm{lm}_{\prec^{(h)}_{(u,v)}}(p) = \mathrm{lm}_{\prec_{(u,v)}}(p_{|h=1}) = \mathrm{lm}_{\prec}(\mathrm{in}_{(u,v)}(p_{|h=1}))$$

by definition of these orderings. Let $f' \in \mathrm{in}_{(u,v)}(I)$ be (u,v)-homogeneous. Then there exists some $f \in I$ such that $f' = \mathrm{in}_{(u,v)}(f)$. Since $G^{(h)}$ is a Gröbner basis with respect to $\prec^{(h)}_{(u,v)}$, there also exists some (ζ,η)-homogeneous $g \in G^{(h)}$ satisfying

$$\mathrm{lm}_{\prec}(\mathrm{in}_{(u,v)}(g)) = \mathrm{lm}_{\prec^{(h)}_{(u,v)}}(g) \mid \mathrm{lm}_{\prec^{(h)}_{(u,v)}}(H_{(\zeta,\eta)}(f)) = \mathrm{lm}_{\prec}(\mathrm{in}_{(u,v)}(f)) = \mathrm{lm}_{\prec}(f'),$$

which concludes the proof. □

Note that $G^{(h)} \subseteq D^{(h)}_{n,(\zeta,\eta)}$, $G^{(h)}|_{h=1} \subseteq D_n$ and $\mathrm{in}_{(u,v)}(G^{(h)}|_{h=1}) \subseteq \mathrm{gr}_{(u,v)}(D_n)$.
Summarizing the results from this section, we obtain the following algorithm to compute the initial ideal.

Algorithm 2.32 (`initialIdeal`)**.**

Input: $I \subseteq D_n$ an ideal, $\prec$ a global ordering on D_n, $0 \neq (u,v) \in \mathbb{R}^{2n}$ a weight vector, $\zeta,\eta \in \mathbb{R}^n_{>0}$ homogenization weights

Output: A Gröbner basis of $\mathrm{in}_{(u,v)}(I)$ with respect to $\prec$

$\prec^{(h)}_{(u,v)}$:= the homogenized ordering as defined in Theorem 2.31
$G^{(h)}$:= a Gröbner basis of $H_{(\zeta,\eta)}(I)$ with respect to $\prec^{(h)}_{(u,v)}$ $\quad \subseteq D^{(h)}_{n,(\zeta,\eta)}$
$G := G^{(h)}|_{h=1}$ $\quad \subseteq D_n$
return $\mathrm{in}_{(u,v)}(G)$ $\quad \subseteq \mathrm{gr}_{(u,v)}(D_n)$

Note that we do not compute a *standard basis* (see e. g. [GP08]) in the algorithm above since we do not work in *localizations*, but rather a *Gröbner basis with respect to a weight vector* as it is called in [SST00]. It is true though, that a Gröbner basis with respect to a weight vector of the form $(-w, w)$ is a (non-reduced) standard basis of an ideal. However, the converse is not true, since computing with respect to the weight $(-w, w)$, we do not compute in any localization, i. e. the units in the ring are still only constants (cf. Corollary 1.31) and not all elements of the form $1 + p$, where $\mathrm{lm}(p) \prec 1$, as in the localized ring.

Example 2.33 (Continuation of Example 1.37). We compute the initial ideal of $I = \langle 3x^2\partial_y + 2y\partial_x, 2x\partial_x + 3y\partial_y + 6\rangle \subseteq D_2$ with respect to the weight vectors $(-1, 0, 1, 0), (0, -1, 0, 1)$ and $(-2, -3, 2, 3)$. We use the degree reverse lexicographical ordering.

```
LIB "bfun.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def D_2 = Weyl(); setring D_2;
ideal I = 3*x^2*Dy+2*y*Dx, 2*x*Dx+3*y*Dy+6;
intvec w1 = 1,0; initialIdealW(I,-w1,w1);
==>_[1]=y*Dx
==>_[2]=2*x*Dx+3*y*Dy+6
==>_[3]=y^2*Dy+2*y
intvec w2 = 0,1; initialIdealW(I,-w2,w2);
==>_[1]=2*x*Dx+3*y*Dy+6
==>_[2]=x^2*Dy
==>_[3]=3*x*y*Dy^2+5*x*Dy
==>_[4]=9*y^2*Dy^3+45*y*Dy^2+35*Dy
intvec w3 = 2,3; initialIdealW(I,-w3,w3);
==>_[1]=2*x*Dx+3*y*Dy+6
==>_[2]=3*x^2*Dy+2*y*Dx
==>_[3]=9*x*y*Dy^2-4*y*Dx^2+15*x*Dy
==>_[4]=27*y^2*Dy^3+8*y*Dx^3+135*y*Dy^2+105*Dy
```

3 Intersecting an ideal with a subalgebra

The main goal of this chapter is to analyze the problem of intersecting an ideal with a subalgebra, which is needed for the computation of global b-functions. We will examine three distinguished approaches.

3.1 Classical elimination

For this section, let $A = \mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij}, 1 \leq i < j \leq n\}\rangle$ be a G-algebra.

Definition 3.1. We call the subalgebra $B \subseteq A$ generated by $x_{i_1}, \ldots, x_{i_r}$, $1 \leq i_1 < \ldots < i_r \leq n, r \geq 1$, an *admissible subalgebra*, if $d_{i_j,i_k}, i_1 \leq i_j < i_k \leq i_r$, are polynomials in $x_{i_1}, \ldots, x_{i_r}$.

Clearly, admissible subalgebras are G-algebras as well.

Definition 3.2. Let $B = \mathbb{K}\langle x_{r+1}, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij}, r+1 \leq i < j \leq n\}\rangle \subseteq A$ be an admissible subalgebra of A. A global ordering $\prec$ on A, which satisfies the ordering condition in the definition of a G-algebra (Definition 1.7) is called an *elimination ordering* for $x_1, \ldots, x_r$, if $\operatorname{lm}(f) \in B$ implies $f \in B$ for any $f \in A$. Moreover, if in this situation $x_1, \ldots, x_r$ generate an admissible subalgebra $C \subseteq A$, we call $\prec$ an elimination ordering for C.

Lemma 3.3 (Elimination Lemma [Lev05]). Let $J \subseteq A$ be an ideal, $B = \mathbb{K}\langle x_{r+1}, \ldots, x_n \mid x_j x_i = c_{ij} x_i x_j + d_{ij}\rangle$ an admissible subalgebra of A and $\prec$ an elimination ordering for $x_1, \ldots, x_r$ on A. If G is a Gröbner basis of J with respect to $\prec$, then $G \cap B$ is a Gröbner basis of $J \cap B$.

Proof. Let $f \in J \cap B$. Then there exists some $g \in G$ such that $\operatorname{lm}(g) \mid \operatorname{lm}(f) \in B$. This implies that $\operatorname{lm}(g) \in B$. Since $\prec$ is an elimination ordering, $g \in B$ holds and thus, the claim follows. □

Therefore, intersections we are interested in can be computed by computing a Gröbner basis with respect to an appropriate elimination ordering.

Elimination orderings, though very useful in theory, also bear disadvantages. The computation of a Gröbner basis with respect to an elimination ordering can be very expensive, both in time and memory consumption.
Another crucial problem is that elimination orderings do not need to exist, in contrast to the commutative case, where the lexicographical ordering for instance is always an elimination ordering.

Example 3.4 ([Lev06]). Let $A = \mathbb{K}\langle x, y \mid \{yx = xy + y^2\}\rangle$ be a G-algebra. Any elimination ordering $\prec$ for y requires $x \prec y$. But this implies $xy \prec y^2$, which contradicts the ordering condition of Definition 1.7, stating that $y^2 \prec yx$ must hold for any ordering $\prec$ on A.
One possibility would be to consider A as a $\mathbb{K}$-algebra equipped with an ordering satisfying $x \prec y$ modulo the two-sided ideal $R := {}_A\langle y^2 - yx + xy\rangle_A$. But the two-sided Gröbner basis of R is infinite, hence doing the elimination via passing to this $\mathbb{K}$-algebra setting is problematic.

3.2 Intersection via preimages

Recall the algorithm for computing the preimage of a left ideal under a homomorphism of G-algebras by [Lev06].

Theorem 3.5 (Preimage of a Left Ideal [Lev06]). Let A, B be G-algebras of Lie type, generated by $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ respectively, subject to finite sets of relations R_A, R_B as in Definition 1.7. Let $\phi : A \to B$ be a homomorphism of $\mathbb{K}$-algebras. Consider the (A, A)-bimodule

$$I_\phi := {}_A\langle\{x_i - \phi(x_i) \mid 1 \le i \le n\}\rangle_A \subseteq A \otimes_{\mathbb{K}} B.$$

Suppose, that there exists an elimination ordering $\prec$ for B on $A \otimes_{\mathbb{K}} B$, satisfying

$$\operatorname{lm}(y_j\phi(x_i) - \phi(x_i)y_j) \prec x_iy_j \quad \text{for } 1 \le i \le n, 1 \le j \le m.$$

Then there are the following statements.

(a) Define $A \otimes_{\mathbb{K}}^{\phi} B$ to be the $\mathbb{K}$-algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_m$ subject to the finite set of relations composed of R_A, R_B and $\{y_jx_i - x_iy_j - y_j\phi(x_i) + \phi(x_i)y_j\}$. Then $A \otimes_{\mathbb{K}}^{\phi} B$ is a G-algebra of Lie type.

(b) Let $J \subseteq B$ be a left ideal, then

$$\phi^{-1}(J) = (I_\phi + J) \cap A \;\subseteq\; A \otimes_{\mathbb{K}}^{\phi} B \cap A.$$

Moreover, this computation can be done by means of elimination.

Remark 3.6. Let A be a G-algebra of Lie type, $s = \sum_{i=1}^{n} a_im_i \in A$, where $a_i \in \mathbb{K}$ and m_i are standard monomials of A.

(a) Consider the homomorphism of $\mathbb{K}$-algebras

$$\phi : \mathbb{K}[s] \to A, s \mapsto \sum_{i=1}^{n} a_i m_i.$$

Then $I \cap \mathbb{K}[s] = \phi^{-1}(I)$.

(b) Additionally, if the standard monomials m_i are pairwise commuting in A, one can break down the computation into two steps. Consider the homomorphisms of $\mathbb{K}$-algebras

$$\begin{aligned} &\phi_1 : \mathbb{K}[m_1, \ldots, m_n] \to A, m_i \mapsto m_i \qquad \text{and} \\ &\phi_2 : \mathbb{K}[s] \to \mathbb{K}[m_1, \ldots, m_n], s \mapsto \sum_{i=1}^{n} a_i m_i. \end{aligned}$$

Then $I \cap \mathbb{K}[s] = \phi_2^{-1}(\phi_1^{-1}(I))$.

By Theorem 3.5 the preimages in both cases can be computed by means of elimination. Moreover, the preimage of ϕ in (a) as well as the preimage of ϕ_2 in (b) can also be computed with the method of principal intersection, which we will describe in the following section.

3.3 The method of principal intersection

The goal of this section is to give an elimination-free alternative for our intersecting problem in the case that the subalgebra is generated by a single element as it is the case for the computation of the global b-function. However, we will consider a much more general setting.
Let A be an associative $\mathbb{K}$-algebra. We assume that A does not contain left or right zero divisors. We are interested in computing the intersection of a left ideal $J \subseteq A$ with a subalgebra $S \subseteq A$, which is generated by an arbitrary element $s \in A \setminus \mathbb{K}$.
Suppose s is *algebraic* over $\mathbb{K}$, i. e. there exists a univariate polynomial $p \in \mathbb{K}[\sigma]$ such that $p(s) = 0$. Let p be of minimal degree with that property. Then S can be viewed as $\mathbb{K}[\sigma]/\langle p \rangle$ and the intersection $J \cap S$ is an ideal in $\mathbb{K}[\sigma]/\langle p \rangle$, hence it is either $\{0\}$ or it is generated by a divisor of p.
From now on we assume that s is not algebraic (i. e. *transcendental*) over $\mathbb{K}$. Then the subalgebra S is isomorphic to the univariate polynomial ring $\mathbb{K}[s]$.
Since $\mathbb{K}[s]$ is a principal ideal domain, the intersection $J \cap S$ is always generated by one element. In other words, we would like to find the monic polynomial $b \in A$ satisfying

$$\langle b \rangle = J \cap \mathbb{K}[s].$$

For this section, we will make two final assumptions, namely that there is a concept of Gröbner bases on A with a corresponding notion of normal forms such that the claim

of Lemma 1.19 holds true and secondly, that there is a monomial ordering $\prec$ on A such that J has a finite left Gröbner basis G with respect to $\prec$.
Then we can distinguish between the following situations:

1. No leading monomials of elements in G divide the leading monomial of any power of s.

2. There is an element in G, whose leading monomial divides the leading monomial of some power of s. In this situation, we have the following sub-cases.

   2.1. $J \cdot s \subseteq J$ and $\dim_{\mathbb{K}}(\mathrm{End}_A(A/J)) < \infty$.

   2.2. One of the two conditions in 2.1. does not hold.

We start with the first case.

Lemma 3.7. If there exists no $g \in G$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(s^k)$ for some $k \in \mathbb{N}_0$, then $J \cap \mathbb{K}[s] = \{0\}$.

Proof. Let $0 \neq b \in J \cap \mathbb{K}[s]$. Then $\mathrm{lm}(b) = \mathrm{lm}(s^k)$ for some $k \in \mathbb{N}_0$. Since $b \in J$, there exists $g \in G$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(b) = \mathrm{lm}(s^k)$. □

In the second situation however, we cannot in general state whether the intersection is trivial or not as the following example illustrates.

Remark 3.8. The converse of the previous lemma does not hold. For instance, consider $\mathbb{K}[x, y]$ and $J = \langle y^2 + x \rangle$. Then $J \cap \mathbb{K}[y] = \{0\}$ while $\{y^2 + x\}$ is a Gröbner basis of J for any ordering.

In situation 2.1. though, the intersection cannot be zero as the following theorem shows, inspired by the sketch of the proof of Theorem 3.11 below in [SST00].

Theorem 3.9. Let $J \cdot s \subseteq J$ and $\dim_{\mathbb{K}}(\mathrm{End}_A(A/J)) < \infty$. Then $J \cap \mathbb{K}[s] \neq \{0\}$.

Proof. Consider the right multiplication with s as a map $A/J \to A/J$ which is a well-defined endomorphism of the A-module A/J as $a - a' \in J$ implies that $(a - a')s \in J \cdot s \subseteq J$, which holds by assumption for all $a, a' \in A$. Since $\mathrm{End}_A(A/J)$ is finite dimensional, linear algebra guarantees that this endomorphism has a well-defined non-zero minimal polynomial μ. Moreover, μ is precisely the monic generator of $J \cap \mathbb{K}[s]$ as $\mu(s) = [0]$ in A/J, hence $\mu(s) \in J \cap \mathbb{K}[s]$, and $\deg(\mu)$ is minimal by definition of the minimal polynomial. □

Remark 3.10. In particular, $\dim_{\mathbb{K}}(\mathrm{End}_A(A/J))$ is finite if A/J itself is a finite dimensional A-module. In the case where A is a Weyl algebra and J is holonomic, we know that $\dim_{\mathbb{K}}(\mathrm{End}_A(A/J)) < \infty$ holds (e. g. [SST00]).
The condition $J \cdot s \subseteq J$ is fulfilled, if s is *central*, i. e. s commutes with all $a \in A$. More specifically, it holds, if s commutes with all $j \in J$, i. e. s lies in the *centralizer* $C_A(J) := \{a \in A \mid aj = ja \text{ for all } j \in J\}$ of J.

By the proof of the theorem, we have reduced our problem of intersecting an ideal with a principal subalgebra to a problem from linear algebra, namely to the one of finding the minimal polynomial of an endomorphism.

Theorem 3.11. The global b-function $b_{I,w}$ of a holonomic ideal $I \subseteq D_n$ is not the zero polynomial for any weight vector $0 \neq w \in \mathbb{R}^n_{\geq 0}$.

Proof. Let $J := \mathrm{in}_{(-w,w)}(I)$ and $s := \sum_{i=1}^n w_i x_i \partial_i$. Recall from Definition 1.36 that $b_{I,w}$ is defined by ${}_{\mathbb{K}[s]}\langle b_{I,w} \rangle = J \cap \mathbb{K}[s]$.
Without loss of generality let $0 \neq p = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta \in J$ be $(-w,w)$-homogeneous. Then we obtain for every monomial in p by using the Leibniz rule (Theorem 1.29)

$$\begin{aligned} x^\alpha \partial^\beta x_i \partial_i &= x^{\alpha+e_i} \partial^{\beta+e_i} + \beta_i x^\alpha \partial^\beta \\ &= (\partial_i x_i^{\alpha_i+1} - (\alpha_i+1) x_i^{\alpha_i}) \frac{x^\alpha}{x_i^{\alpha_i}} \partial^\beta + \beta_i x^\alpha \partial^\beta \\ &= (\partial_i x_i - (\alpha_i+1) + \beta_i) x^\alpha \partial^\beta \\ &= (x_i \partial_i - \alpha_i + \beta_i) x^\alpha \partial^\beta. \end{aligned}$$

Put $m := \deg_{(-w,w)}(p)$. Since p is $(-w,w)$-homogeneous, $m = -w\alpha + w\beta$ for each non-zero term $c_{\alpha,\beta} x^\alpha \partial^\beta$ of p. Hence,

$$\begin{aligned} p \cdot s = p \sum_{i=1}^n w_i x_i \partial_i &= \sum_{i=1}^n w_i p x_i \partial_i = \sum_{i=1}^n w_i \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta x_i \partial_i \\ &= \sum_{i=1}^n w_i \sum_{\alpha,\beta} (x_i \partial_i - \alpha_i + \beta_i) c_{\alpha,\beta} x^\alpha \partial^\beta \\ &= (\sum_{i=1}^n w_i x_i \partial_i)(\sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta) + \sum_{i=1}^n w_i \sum_{\alpha,\beta} (-\alpha_i + \beta_i) c_{\alpha,\beta} x^\alpha \partial^\beta \\ &= s \cdot p + \sum_{\alpha,\beta} (\sum_{i=1}^n (-w_i \alpha_i + w_i \beta_i)) c_{\alpha,\beta} x^\alpha \partial^\beta \\ &= s \cdot p + \sum_{\alpha,\beta} (-w\alpha + w\beta) c_{\alpha,\beta} x^\alpha \partial^\beta \\ &= s \cdot p + m \cdot p = (s+m) \cdot p \in J. \end{aligned}$$

Therefore, $J \cdot s \subseteq J$ holds. Since D/J is holonomic (cf. [SST00, Theorem 2.2.1]), Remark 3.10 and Theorem 3.9 yield the claim. □

If one knows in advance that the intersection is not zero, the following algorithm can be used for computing its monic generator. Recall that we assume the existence of a finite left Gröbner basis for the ideal $J \subseteq A$.

Algorithm 3.12 (`principalIntersect`).
Input: $s \in A, J \subseteq A$ a left ideal such that $J \cap \mathbb{K}[s] \neq \{0\}$.
Output: $b \in \mathbb{K}[s]$ monic such that $J \cap \mathbb{K}[s] = \langle b \rangle$
  $G :=$ a finite left Gröbner basis of J
  $i := 1$
  loop
    if there exist $a_0, \dots, a_{i-1} \in \mathbb{K}$ such that $\mathrm{NF}(s^i, G) + \sum_{j=0}^{i-1} a_j \, \mathrm{NF}(s^j, G) = 0$ **then**
      return $b := s^i + \sum_{j=0}^{i-1} a_j s^j$
    else
      $i := i + 1$
    end if
  end loop

Note that because $0 = \mathrm{NF}(s^i, G) + \sum_{j=0}^{i-1} a_j \, \mathrm{NF}(s^j, G) = \mathrm{NF}(s^i + \sum_{j=0}^{i-1} a_j s^j, G)$ (Lemma 1.19(c)) is equivalent to $s^i + \sum_{j=0}^{i-1} a_j s^j \in J$ (Lemma 1.19(a)), the algorithm searches for a monic polynomial in $\mathbb{K}[s]$ that also lies in J. This is done by going degree by degree through the powers of s until there is a linear dependency. This approach also ensures the minimality of the degree of the output. The algorithm terminates if and only if $J \cap \mathbb{K}[s] \neq \{0\}$. Note that this approach does not require $\mathbb{K}$ to be of characteristic zero.

The check whether there is a linear dependency over $\mathbb{K}$ between the computed normal forms of the powers of s can be done by the following algorithm, which carries the concept of Gaussian elimination to polynomials.

Algorithm 3.13 (`linReduce`).
Input: $f \in A$ a polynomial, $\{f_1, \dots, f_k\} \subseteq A$ a subset
Output: $a \in \mathbb{K}^k, p \in A$ such that $p = f - \sum_{i=1}^k a_i f_i$ and $\mathrm{lm}(p) \neq \mathrm{lm}(f_i)$ for $1 \leq i \leq k$
  $a := 0 \in \mathbb{K}^k$
  $p := f$
  while there exists $i \in \{1, \dots k\}$, such that $\mathrm{lm}(p) = \mathrm{lm}(f_i)$ **do**

    $p := p - \frac{\mathrm{lc}(p)}{\mathrm{lc}(f_i)} f_i$
    $a_i := a_i - \frac{\mathrm{lc}(f)}{\mathrm{lc}(f_i)}$

  end while
  return a, p

This algorithm computes a "linear reductum" of a polynomial, i. e. no monomial multiplications are being used. Here, only leading monomials are compared and (if possible) reduced by linear operations, making use of the fact that there is a linear dependency between the polynomials if and only if there is a linear dependency between the leading monomials of the linear reducta.

3.3.1 Enhanced computation of normal forms

When computing normal forms of the form $\mathrm{NF}(s^i, G)$ like in Algorithm 3.12 we can speed up the reduction process by making use of the previously computed normal forms. Let G be a finite Gröbner basis of the ideal $J \subseteq A$ and let $f \in J$. Then we have for all $a \in A$ by using the linearity of the normal form (Lemma 1.19(c))

$$\begin{aligned}\mathrm{NF}([f,a],G) &= \mathrm{NF}(fa - af, G) = \mathrm{NF}(fa,G) - \mathrm{NF}(af,G) = \mathrm{NF}(fa,G) - 0 \\ &= \mathrm{NF}(fa,G),\end{aligned}$$

since $af \in J$ and Lemma 1.19(a). This means, we can immediately erase all terms of f commuting with a.

Lemma 3.14. Let $f \in A$. For $i \in \mathbb{N}$ denote $r_i := \mathrm{NF}(f^i, J)$, $q_i := f^i - r_i \in J$ and $c_i := \frac{\mathrm{lc}(q_i r_1)}{\mathrm{lc}(r_1 q_i)}$ provided $r_1 q_i \neq 0$. For $r_1 q_i = 0$ we put $c_i := 0$. Then we have for all $i \in \mathbb{N}$

$$r_{i+1} = \mathrm{NF}(fr_i, J) = \mathrm{NF}([f^i - r_i, r_1]_{c_i} + r_i r_1, J),$$

where $[a,b]_c := ab - c \cdot ba$ denotes the *skew Lie bracket* for $a, b \in A, c \in \mathbb{K} \setminus \{0\}$.

Proof. It holds that $f^{i+1} = fq_i + fr_i$ reduces to fr_i, which shows the first equation. On the other hand,

$$f^{i+1} = q_i f + r_i f = q_i(q_1 + r_1) + r_i(q_1 + r_1) = q_i q_1 + q_i r_1 + r_i q_1 + r_i r_1$$

reduces to $q_i r_1 + r_i r_1 = (f^i - r_i) r_1 + r_i r_1$, which again reduces to $[f^i - r_i, r_1]_{c_i} + r_i r_1$, proving the second equation. □

As a direct consequence, we obtain the following result for some $\mathbb{K}$-algebras of special importance.

Corollary 3.15. If A is a G-algebra of Lie type (e. g. a Weyl algebra), then

$$r_{i+1} = \mathrm{NF}(fr_i, J) = \mathrm{NF}([f^i - r_i, r_1] + r_i r_1, J) \text{ holds.}$$

If A is commutative, we have

$$r_{i+1} = \mathrm{NF}(r_i r_1, J) = \mathrm{NF}(r_1^{i+1}, J).$$

The lemma and the corollary, respectively, are of utter utility in practice. See Section 6.2.5 for the remarkable impact in computations.
Note, that computing the Lie bracket $[f, g]$ by making use of the properties given in Lemma 1.22 in theory as well as in practice is easier and faster than to compute $f \cdot g - g \cdot f$, see e. g. [LS03].

3.4 Applications

Apart from computing global b-functions, there are various other applications of Algorithm 3.12, which we address in this section.

3.4.1 Computing the global b-function of an ideal

We now have gathered all means necessary to compute the global b-function of an ideal.

Algorithm 3.16 (`bfctIdeal`).
Input: $I \subseteq D_n$ a holonomic ideal, $0 \neq w \in \mathbb{R}^n_{>0}$ a weight vector
Output: the global b-function $b_{I,w}(s) \in \mathbb{K}[s]$ of I with respect to w
  $J :=$ `initialIdeal`$(I, (-w, w))$ $\quad\rightarrow$ Algorithm 2.32
  $s := \sum_{i=1}^{n} w_i x_i \partial_i$
  return $b_{I,w}(s) :=$ `principalIntersect`(s, J) $\quad\rightarrow$ Algorithm 3.12

Example 3.17 (Continuation of Example 2.33).
For $I := \langle 3x^2\partial_y + 2y\partial_x, 2x\partial_x + 3y\partial_y + 6\rangle \subseteq D_2$ we compute

$$\mathrm{in}_{(-w^{(i)},w^{(i)})}(I) \cap \mathbb{K}[w_1^{(i)} x\partial_x + w_2^{(i)} y\partial_y],$$

where $w^{(1)} := (1, 0), w^{(2)} := (0, 1)$ and $w^{(3)} := (2, 3)$.

```
LIB "bfun.lib";
ring r = 0,(x,y,Dx,Dy),dp;
def D_2 = Weyl(); setring D_2;
ideal I = 3*x^2*Dy+2*y*Dx, 2*x*Dx+3*y*Dy+6;
intvec w1 = 1,0; ideal J1 = initialIdealW(I,-w1,w1); poly s1 = x*Dx;
vector v1 = pIntersect(s1,J1); v1;
==> gen(3)+3/2*gen(2)
```

The procedure `pIntersect` returns an object of the type `vector` for technical reasons. Here, `gen(i)` stands for s^{i-1}. So the result is $s^2 + \frac{3}{2}s$. We convert `v1` to a polynomial and factorize it. The result is the list of the roots and their corresponding multiplicities.

```
bFactor(vec2poly(v1));
==> [1]:
==>    _[1]=0
==>    _[2]=-3/2
==> [2]:
==>    1,1
```

We proceed the same way in the other computations.

```
intvec w2 = 0,1; ideal J2 = initialIdealW(I,-w2,w2); poly s2 = y*Dy;
bFactor(vec2poly(pIntersect(s2,J2)));
==> [1]:
==>    _[1]=0
==>    _[2]=-2/3
==>    _[3]=-4/3
==> [2]:
==>    1,1,1
intvec w3 = 2,3; ideal J3 = initialIdealW(I,-w3,w3);
poly s3 = 2*x*Dx+3*y*Dy; bFactor(vec2poly(pIntersect(s3,J3)));
==> [1]:
==>    _[1]=-6
==> [2]:
==>    1
```

Hence,

$$b_{I,w^{(1)}} = s(s+\frac{3}{2}), \quad b_{I,w^{(2)}} = s(s+\frac{2}{3})(s+\frac{4}{3}) \quad \text{and} \quad b_{I,w^{(3)}} = s+6.$$

3.4.2 Solving zero-dimensional systems

Recall that one of the original motivations for the development of Gröbner bases was to solve *zero-dimensional systems.*

Definition 3.18. A proper ideal $I \subseteq \mathbb{K}[x_1,\ldots,x_n]$ is called *zero-dimensional*, if the quotient $\mathbb{K}[x_1,\ldots,x_n]/I$ is finite dimensional as a $\mathbb{K}$-vector space.

Lemma 3.19. A proper ideal I is zero-dimensional if and only if there exist $0 \neq f_i \in I \cap \mathbb{K}[x_i]$ for each $1 \leq i \leq n$.

Proof. Clearly, $I \cdot x_i = x_i \cdot I = I$. If $\dim_{\mathbb{K}}(\mathbb{K}[x_1,\ldots,x_n]/I)$ is finite, then so is $\dim_{\mathbb{K}}(\mathrm{End}_{\mathbb{K}}(\mathbb{K}[x_1,\ldots,x_n]/I))$. Then Theorem 3.9 states that $I \cap \mathbb{K}[x_i] \neq \{0\}$.
Conversely, let $0 \neq f_i \in I \cap \mathbb{K}[x_i]$ for each $1 \leq i \leq n$. Without loss of generality, we may choose f_i of minimal degree d_i. It follows that $[1],[x_i],[x_i^2],\ldots,[x_i^{d_i}]$ are $\mathbb{K}$-linearly dependent in $\mathbb{K}[x_1,\ldots,x_n]/I$. Hence, there exist $0 < e_i \leq d_i$ such that $\{[x_1^{i_1}\cdots x_n^{i_n}] \mid 0 \leq i_j \leq e_j\}$ is a $\mathbb{K}$-basis of $\mathbb{K}[x_1,\ldots,x_n]/I$ and thus, $\dim_{\mathbb{K}}(\mathbb{K}[x_1,\ldots,x_n]/I) \leq \prod_{i=1}^n d_i < \infty$, i. e. I is zero-dimensional. $\square$

Corollary 3.20. Let $I \subseteq \mathbb{K}[x_1,\ldots,x_n]$ be zero-dimensional. Then

$$I_{|x_i=a} \subseteq \mathbb{K}[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]$$

is zero-dimensional for all $a \in \mathbb{K}$.

Proof. By the previous lemma, there exists $0 \neq f_j \in I \cap \mathbb{K}[x_j]$ for all $1 \leq j \leq n$. Using $f_j = f_{j_{|x_i=a}} \in I_{|x_i=a} \cap \mathbb{K}[x_j]$ for all $j \neq i$, the claim follows again by the previous lemma. □

Corollary 3.21. Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be zero-dimensional. If the zero-set $\mathcal{V}(I)$ of I is non-empty, then its cardinality is finite.

Proof. A univariate polynomial $f \neq 0$ has only finitely many roots. □

Algorithm 3.22 (`solveZeroDimSystem`).
Input: $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ a zero-dimensional ideal
Output: $\mathcal{V}(I)$
  $f \in \mathbb{K}[x_1]$ such that $\langle f \rangle = I \cap \mathbb{K}[x_1]$ → Algorithm 3.12
  for $a \in \mathcal{V}(f)$ **do**
    $I(a) := I_{|x_1=a} \subseteq \mathbb{K}[x_2, \ldots, x_n]$
    $V_a :=$ `solveZeroDimSystem`$(I(a))$
  end for
  return $V := \{(a, a') \mid a \in \mathcal{V}(f), a' \in V_a\}$

The computation of the principal ideals in the previous algorithm can be done with `pIntersect` (Algorithm 3.12).
The advantage of using Algorithm 3.22 instead of the classic triangularization techniques, lies in the avoidance of computing a Gröbner basis with respect to a lexicographic ordering or a more general elimination ordering, which can be very hard. The price we need to pay is the performance of multiple Gröbner basis computations – but we may freely choose any, hence better suited, ordering.
A similar approach is used in the celebrated FGLM algorithm [FGLM93]. See also [NY99] for a different method.

3.4.3 Computing central characters

Let A be a G-algebra. The *center* of A,

$$Z(A) := C_A(A) = \{a \in A \mid [a, b] = 0 \text{ for all } b \in A\},$$

is isomorphic to a commutative polynomial ring. The intersection of a left ideal with $Z(A)$ is important for many algorithms, among other for the computation of the *central character decomposition* of a finitely presented module [Lev05]. In the situation, where the center of A is generated by one element, we can apply Algorithm 3.12 to compute the intersection, which is known to be often quite nontrivial, without engaging much more expensive Gröbner basis computations, which use elimination.

Example 3.23. Consider the *universal enveloping algebra* of the *Lie algebra* $\mathfrak{sl}_2$, $U := U(\mathfrak{sl}_2, \mathbb{K}) := \mathbb{K}\langle e, f, h \mid [e, f] = h, [h, e] = 2e, [h, f] = -2f\rangle$. It is known, that over a field of characteristic zero, the center of U equals $Z(U) = \mathbb{K}[4ef + h^2 - 2h]$. Consider the set $F := \{e^{11}, f^{12}, h^5 - 10h^3 + 9h\} \subseteq U$. Let $L := {}_U\langle F\rangle$ be the left ideal and $T := {}_U\langle F\rangle_U$ the two-sided ideal, both generated by F. Then consider the U-modules $M_L = U/L$ and $M_T = U/T$, which turn out to be finite-dimensional over $\mathbb{K}$. We are interested in intersecting L and T with $Z(U)$ and factorizing the output polynomial in one variable.

```
LIB "ncalg.lib"; LIB "central.lib"; LIB "bfun.lib";
def U = makeUsl(2); setring U;              // U = U(sl_2, Q)
poly z = center(2)[1];                      // generator of the center Z(U)
ideal F = e^11,f^12,(h-3)*(h-1)*h*(h+1)*(h+3);
ideal L = std(F);                           // left Gröbner basis of L
vdim(L);                                    // dim_K(U/L)
==> 559
vector vL = pIntersect(z,L);                // L ∩ K[z]
ideal T = twostd(I);                        // two-sided Gröbner basis of T
vdim(T);                                    // dim_K(U/T)
==> 21
vector vT = pIntersect(z,T);                // T ∩ K[z]
ring r = 0,z,dp;                            // commutative univariate ring
// pretty-print factorization of polynomials:
print(matrix(factorize(vec2poly(imap(U,vL)),1)));         // for L∩K[z]
==> z-3,z,z-440,z-8,z-48,z-168,z-15,z-99,z-120,z-255,z-483,z-575,z+1,
    z-399,z-143,z-195,z-63,z-80,z-288,z-360,z-224,z-323,z-35,z-24
print(matrix(factorize(vec2poly(imap(U,vT)),1)));         // for T∩K[z]
==> z-3,z,z-15
```

Notably, all the computations, thanks to Algorithm 3.12, were completed in a couple of seconds, while the Gröbner-driven elimination approach ran out of memory after half an hour.

3.5 Intersecting an ideal with a multivariate subalgebra

We now consider the case where we intersect J with the subalgebra $\mathbb{K}[s] = \mathbb{K}[s_1, \ldots, s_r]$ of an associative $\mathbb{K}$-algebra A for non-constant, pairwise commuting $s_1, \ldots, s_r \in A$. The following result is a direct consequence of Lemma 3.19.

Corollary 3.24. The ideal $J \cap \mathbb{K}[s]$ is zero-dimensional (in $\mathbb{K}[s]$) if and only if for all $1 \le i \le r$ there exist $f_i \in J$ such that $\mathrm{lm}(f_i) = s_i^{d_i}$ for some $d_i \in \mathbb{N}_0$.

Lemma 3.25. For a finite left Gröbner basis G of J,

$$\begin{aligned}\mathrm{GKdim}(\mathbb{K}[s]) &\geq \mathrm{GKdim}(\mathbb{K}[s]/(J \cap \mathbb{K}[s])) \\ &\geq \mathrm{GKdim}(\mathbb{K}[s]/(L(G) \cap \mathbb{K}[s])).\end{aligned}$$

Proof. For all $f \in J \cap \mathbb{K}[s]$ there exists $g \in G$ such that $\mathrm{lm}(g) \mid \mathrm{lm}(f)$, which implies $\mathrm{lm}(g) \in \mathbb{K}[s]$ and thus, the claim follows. □

Note that the first inequality is strict if and only if $J \cap \mathbb{K}[s] \neq \{0\}$.
We give a generalization of Algorithm 3.12 to compute a partial Gröbner basis of $J \cap \mathbb{K}[s]$ up to a specified bound $k \in \mathbb{N}$. Here, a *partial Gröbner basis* G' of an ideal I is a subset of a Gröbner basis of I such that G' is a Gröbner basis of $\langle G' \rangle$.

Algorithm 3.26 (`intersectUpTo`).
Input: $s_1, \ldots, s_r \in A$ pairwise commuting, $J \subseteq A$ a left ideal, $k \in \mathbb{N}$ an upper degree bound
Output: a GB for $J \cap \mathbb{K}[s_1, \ldots, s_r]$ up to degree k
  $G :=$ a partial left Gröbner basis of J consisting of elements up to degree k
  $d := 0$
  $B := \emptyset$
  while $d \leq k$ **do**
    $M_d := \{s^\alpha \mid |\alpha| \leq d\}$
    if there exist $a_m \in \mathbb{K}$, not all 0, such that $\sum_{m \in M_d} a_m \, \mathrm{NF}(m, G) = 0$ **then**
      if $\sum_{m \in M_d} a_m m \notin \langle B \rangle$ **then**
        $B := B \cup \{\sum_{m \in M_d} a_m m\}$
      end if
    end if
    $d := d + 1$
  end while
  return B

A couple of improvements can be made to speed up the computation time.
If $p \in B$ with $\mathrm{lm}(p) = m$ has been found, any monomial which is a multiple of m can be discarded in the following iterations.
Let G be a Gröbner basis of J with respect to some fixed ordering $\prec$. By using $p \in J \cap \mathbb{K}[s]$ if and only if $\mathrm{lm}(p) \in L(G) \cap \mathbb{K}[s]$, one may disregard $\{m \in M_d \mid \max_\prec\{m' \in L(G) \cap M_d\} \prec m\}$.
Further note that $\mathrm{NF}(m, G) = m$, if $m \notin L(G) \cap \mathbb{K}[s]$.
Using these improvements and choosing $\prec$ to be a degree ordering and the elements in B to be monic, the output of the algorithm equals the reduced Gröbner basis of $J \cap \mathbb{K}[s]$ with respect to $\prec$ up to degree k. However, in general no termination criterion is known to us yet, that is apriori we do not know when we already have the complete needed

basis of the intersection. Nevertheless, the termination is predictable if $J \cap \mathbb{K}[s]$ is a principal ideal in $\mathbb{K}[s]$. This situation often arises in the computation of *Bernstein-Sato ideals*, see [ALMM10, ABL$^+$10]. Moreover, another possibility for the algorithm to stop will be when the set of monomials we consider becomes empty on some step, which is the case if and only if $J \cap \mathbb{K}[s]$ is zero-dimensional.
It is also possible to generalize the results above by replacing the commutativity condition for $s_1, \dots, s_r$ with the condition, that $s_1, \dots, s_r$ generate a subalgebra S of A such that S is a G-algebra.
Note, that under some extra requirements the algorithm will terminate after finitely many steps without setting an explicit degree bound. Hence, in such cases a generally complicated elimination with Gröbner bases can be replaced by much easier and predictable Gröbner-free approach. The latter will, of course, allow to solve harder computational problems.
As noted in Example 3.4, there are algebras where appropriate elimination orderings do not exist. Nevertheless, it is obvious, that the intersection problem in those algebras can still have nontrivial solutions. Hence, Algorithms 3.12 and 3.26 respectively, are indeed the only computational possibilities to get some information about such intersections.

Summarizing, for the computation of the intersection of an ideal and a principal subalgebra, we have the following choices:

1. via Gröbner based elimination working with:
   1.1. classical elimination,
   1.2. the preimage of a left ideal:
      1.2.1. in one step,
      1.2.2. in two steps,
2. the elimination-free method of Principal Intersection.

4 Bernstein-Sato polynomials

In this chapter, we focus on the global b-function of a hypersurface defined by a non-constant element $f \in \mathbb{K}[x_1, \ldots, x_n]$, also known as Bernstein-Sato polynomial. Recall from Definition 1.39 that the Bernstein-Sato polynomial is defined via the global b-function of the Malgrange ideal I_f. We first deal with this approach. Then, after utilizing the crucial mean of the *Gel'fand-Kirillov dimension*, we study a closely related, yet different method to compute the Bernstein-Sato polynomial.

4.1 Applying the global b-function to the Malgrange ideal

As mentioned above, according to its definition (Definition 1.39) the Bernstein-Sato polynomial $b_f(s)$ of a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ can be computed by applying the concept of global b-functions for ideals to the Malgrange ideal of f and a specific choice of the weight vector.
This directly leads to the following algorithm.

Algorithm 4.1 (`bfct`).
Input: $f \in \mathbb{K}[x_1, \ldots, x_n]$
Output: the Bernstein-Sato polynomial $b_f(s) \in \mathbb{K}[s]$ of f
$I_f := \langle t - f, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t, \ldots, \partial_n + \frac{\partial f}{\partial x_n}\partial_t \rangle \subseteq D_n\langle t, \partial_t \rangle$ the Malgrange ideal of f
$w := (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ such that the weight of ∂_t is 1
$b_{I,w}(s) :=$ `bfctIdeal(`I_f, w`)` $\qquad \rightarrow$ Algorithm 3.16
return $b_f(s) := b_{I,w}(-s-1)$

Remark 4.2. In [Nor02], the following choice of homogenization weights is proposed for an efficient Gröbner basis computation:

$$\zeta = (\deg_{\hat{u}}(f), \hat{u}_1, \ldots, \hat{u}_n),$$
$$\eta = (1, \deg_{\hat{u}}(f) - \hat{u}_1 + 1, \ldots, \deg_{\hat{u}}(f) - \hat{u}_n + 1),$$

such that the weight of t is $\deg_{\hat{u}}(f)$, the weight of ∂_t is 1 and $\hat{u} \in \mathbb{R}^n_{>0}$ satisfying $\hat{u}_i < \deg_{\hat{u}}(f) + 1$ is a vector that may be chosen heuristically.

Lemma 4.3. For $f \in \mathbb{K}[x_1, \ldots, x_n]$ let $F = \{t - f, \partial_i + \frac{\partial f}{\partial x_i}\partial_t \mid 1 \leq i \leq n\}$ be the set of generators of the Malgrange ideal I_f and let $\hat{f} := H_{(\zeta,\eta)}(f) \in D^{(h)}_{n,(\zeta,\eta)}$ denote the weighted homogenization of f with respect to (ζ, η) (see Definition 2.26). If (ζ, η) is chosen as in the previous remark,

$$H_{(\zeta,\eta)}(F) = \{t - \hat{f}, \partial_i + \frac{\partial \hat{f}}{\partial x_i}\partial_t \mid 1 \leq i \leq n\}.$$

Proof. Since $f \in \mathbb{K}[x_1, \ldots, x_n]$ does not contain any t, ∂_t or ∂_i, we have

$$\deg_{(\zeta,\eta)}(\hat{f}) = \deg_{(\zeta,\eta)}(f) = \deg_{\hat{u}}(f) = \deg_{(\zeta,\eta)}(t).$$

Further,

$$\deg_{(\zeta,\eta)}(\frac{\partial \hat{f}}{\partial x_i}\partial_t) = \deg_{(\zeta,\eta)}(\frac{\partial \hat{f}}{\partial x_i}) + \deg_{(\zeta,\eta)}(\partial_t) = \deg_{\hat{u}}(f) - \hat{u}_i + 1 = \deg_{(\zeta,\eta)}(\partial_i). \qquad \square$$

Remark 4.4. By the experiments we have performed, we propose to use an ordering for the computation of the initial ideal of I_f as follows. Let $e_i \in \mathbb{R}^n$ denote the i-th standard basis vector. We define a *valuation function*

$$\nu_f : \{x_1, \ldots, x_n\} \to \mathbb{N}_0 : x_i \mapsto \deg_{e_i}(f)$$

and propose to choose an ordering $\prec$ that satisfies $x_i \prec x_j$ if and only if $\nu_f(x_i) \geq \nu_f(x_j)$. That is, "less complex" variables are preferred. See Section 6.2.3 for experimental results.

4.2 The s-parametric annihilator

Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ and let s be a new indeterminate. We consider the $(n+1)$-th Weyl algebra $D_{n+1} \cong D_n\langle t, \partial_t\rangle$ with additional generators t, ∂_t and the commutative ring $R_f := \mathbb{K}[x, s, f^{-1}]$. The free R_f-module $R_f \cdot f^s$ generated by the formal symbol f^s becomes a $D_n\langle t, \partial_t\rangle$-module by the operation

$$\bullet : D_n\langle t, \partial_t\rangle \times R_f \cdot f^s \to R_f \cdot f^s$$

defined by

$$\begin{aligned}
x_i \bullet g(x,s) \cdot f^{s+j} &:= x_i \cdot g(x,s) \cdot f^{s+j}, & 1 \leq i \leq n,\\
\partial_i \bullet g(x,s) \cdot f^{s+j} &:= \frac{\partial g}{\partial x_i} \cdot f^{s+j} + (s+j) \cdot g(x,s) \cdot \frac{\partial f}{\partial x_i} \cdot f^{s+j-1}, & 1 \leq i \leq n,\\
t \bullet g(x,s) \cdot f^{s+j} &:= g(x, s+1) \cdot f^{s+j+1} & \text{and}\\
\partial_t \bullet g(x,s) \cdot f^{s+j} &:= -s \cdot g(x, s-1) \cdot f^{s+j-1}
\end{aligned}$$

for $g \in \mathbb{K}[x, s]$ and $f^{s+j} := f^j \cdot f^s$, $j \in \mathbb{Z}$. That is, x_i acts via multiplication, ∂_i via formal derivation, t via shift and ∂_t via shift and multiplication with $-s$.

Definition 4.5. The *annihilator* of f^s in $D_n\langle t,\partial_t\rangle$ is defined to be

$$\mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s) := \{p \in D_n\langle t,\partial_t\rangle \mid p \bullet f^s = 0\}.$$

It is clear, that $\mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)$ is a left ideal in $D_n\langle t,\partial_t\rangle$, since $f^s \in R\cdot f^s$ and $R\cdot f^s$ is a $D_n\langle t,\partial_t\rangle$-module.
Recall the Malgrange ideal I_f from Definition 1.38. Our first goal in this section is to prove that I_f coincides with $\mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)$. Before we give a proof, we first derive other properties of I_f.

Lemma 4.6. Consider the Malgrange ideal of $f \in \mathbb{K}[x_1,\dots,x_n]$,

$$I_f = \langle t-f, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t, \dots, \partial_n + \frac{\partial f}{\partial x_n}\partial_t\rangle \subseteq D_n\langle t,\partial_t\rangle,$$

and choose a global ordering $\prec$ satisfying

$$\mathrm{lm}(t-f) = t \qquad \text{and} \qquad \mathrm{lm}(\partial_i + \frac{\partial f}{\partial x_i}\partial_t) = \partial_i\ .$$

Then the given generators of I_f build a Gröbner basis with respect to $\prec$.

Proof. We apply Buchberger's Criterion (Theorem 1.21) to the generators. Let $a \to b$ denote the reduction of a to b with respect to I_f. By the Generalized Product Criterion (Lemma 1.23) and by Corollary 1.25, we have

$$\begin{aligned}\mathrm{spoly}(t-f,\partial_i + \frac{\partial f}{\partial x_i}\partial_t) &\to [t-f,\partial_i + \frac{\partial f}{\partial x_i}\partial_t] = t\frac{\partial f}{\partial x_i}\partial_t - \frac{\partial f}{\partial x_i}\partial_t t - f\partial_i + \partial_i f\\ &= (t\partial_t - \partial_t t)\frac{\partial f}{\partial x_i} + (\partial_i f - f\partial_i) = -\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_i} = 0\end{aligned}$$

and

$$\begin{aligned}\mathrm{spoly}(\partial_i + \frac{\partial f}{\partial x_i}\partial_t, \partial_j + \frac{\partial f}{\partial x_j}\partial_t) &\to [\partial_i + \frac{\partial f}{\partial x_i}\partial_t, \partial_j + \frac{\partial f}{\partial x_j}\partial_t]\\ &= \frac{\partial f}{\partial x_i}\partial_t\partial_j - \partial_j\frac{\partial f}{\partial x_i}\partial_t + \partial_i\frac{\partial f}{\partial x_j}\partial_t - \frac{\partial f}{\partial x_j}\partial_t\partial_i\\ &= -\frac{\partial f}{\partial x_i\partial x_j}\partial_t + \frac{\partial f}{\partial x_j\partial x_i}\partial_t = 0.\end{aligned}$$

Hence, the claim follows. □

Theorem 4.7. The Malgrange ideal I_f is holonomic in $D_n\langle t,\partial_t\rangle$.

Proof. By Lemma 4.6, the given generators of I_f already form a Gröbner basis with respect to an appropriate ordering and we have $L(I_f) = \langle t,\partial_1,\dots,\partial_n\rangle$ by Theorem 1.21. Applying Theorem 2.24 yields

$$\begin{aligned}\mathrm{GKdim}(I_f) &= \mathrm{GKdim}(D_n\langle t,\partial_t\rangle/I_f) = \dim(\mathbb{K}[x_1,\dots,x_n,\partial_1,\dots,\partial_n,t,\partial_t]/\langle t,\partial_1,\dots,\partial_n\rangle)\\ &= \dim(\mathbb{K}[x_1,\dots,x_n,\partial_t]) = n+1.\end{aligned}$$

□

Corollary 4.8. The Malgrange ideal I_f is a proper ideal.

Proof. By Theorems 2.20 and 4.7 we have, $\mathrm{GKdim}(I_f) = n+1 < 2n+2 = \mathrm{GKdim}(D_n)$. $\square$

Theorem 4.9. The Malgrange ideal I_f is a maximal left ideal.

Proof. Proceeding similarly to the proof of Theorem 4.7, $L(I_f) = \langle t, \partial_1, \ldots, \partial_n\rangle$ for a suitable ordering $\prec$. Suppose, I_f is not maximal. Then there exists an ideal J containing I_f. Let G be a Gröbner basis of J with respect to $\prec$ such that G contains the generators of I_f, i. e. we add new elements $P := \{p_1, \ldots, p_m\}$, $m \geq 1$, to the generators of I_f such that we obtain a Gröbner basis of J. Without loss of generality, let $\mathrm{lm}(p_i) \in \langle x, \partial_t\rangle$ and let Λ denote the image of $L(G)$ under the isomorphism ψ of vector spaces from Theorem 2.24. Then, Theorems 2.21 and 2.24 imply

$$\begin{aligned}\mathrm{GKdim}(D_n\langle t,\partial_t\rangle/J) &= \mathrm{GKdim}(\mathbb{K}[x,\partial,t,\partial_t]/\Lambda) = \mathrm{GKdim}(\mathbb{K}[x,\partial_t]/L(P)) \\ &< \mathrm{GKdim}(\mathbb{K}[x,\partial_t]) = n+1,\end{aligned}$$

which contradicts Bernstein's inequality (Theorem 2.22). $\square$

Lemma 4.10. The Malgrange ideal I_f coincides with $\mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)$.

Proof. We check the desired property of the generators of I_f:

$$\begin{aligned}(t-f)\bullet f^s &= f^{s+1} - f^{s+1} = 0 \qquad \text{and} \\ (\partial_i + \frac{\partial f}{\partial x_i}\partial_t)\bullet f^s &= s\frac{\partial f}{\partial x_i}f^{s-1} - \frac{\partial f}{\partial x_i}sf^{s-1} = 0.\end{aligned}$$

So we have $I_f \subseteq \mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)$. Since $1\bullet f^s \neq 0$, $\mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)$ is a proper ideal. The maximality of I_f (Theorem 4.9) then implies the claim. $\square$

We now consider the subalgebra $D_n[t\cdot\partial_t] \subseteq D_n\langle t,\partial_t\rangle$ and the G-algebra $D_n[s] := D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$.

Definition 4.11. The *s-parametric annihilator* of f is defined to be the left ideal $\mathrm{Ann}_{D_n[s]}(f^s) := \{p \in D_n[s] \mid p\bullet f^s = 0\}$, where $\bullet$ denotes the action defined at the beginning of this chapter for x and ∂ and moreover, the central element s acts via multiplication.

There is a strong relation between the s-parametric annihilator and the Malgrange ideal I_f as well.

Theorem 4.12. The s-parametric annihilator of f equals $(I_f \cap D_n[t\partial_t])_{|t\partial_t=-s-1}$.

Proof. Consider the isomorphism of the $\mathbb{K}$-algebras $D_n[s]$ and $D_n[t\partial_t]$ induced by the *algebraic Mellin transform*

$$s \mapsto -t\partial_t - 1.$$

The action of $D_n\langle t, \partial_t\rangle$ defined at the beginning of this chapter restricted to its subalgebra $D_n[t\partial_t]$ yields

$$t\partial_t \bullet g(x,s)\cdot f^{s+j} = t \bullet (-s\cdot g(x,s-1)\cdot f^{s+j-1}) = -(s+1)\cdot g(x,s)\cdot f^{s+j}$$

and therefore

$$(-t\partial_t - 1)\bullet g(x,s)\cdot f^{s+j} = (s+1)\cdot g(x,s)\cdot f^{s+j} - g(x,s)\cdot f^{s+j} = s\cdot g(x,s)\cdot f^{s+j}.$$

Thus, the action of $D_n[t\partial_t]$ is compatible with the one defined on $D_n[s]$ and applying the inverse of the algebraic Mellin transform and Lemma 4.10 to the equations

$$\mathrm{Ann}_{D_n[t\partial_t]}(f^s) = \mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)\cap D_n[t\partial_t] = I_f \cap D_n[t\partial_t]$$

concludes the proof. □

Now we return to the Bernstein-Sato polynomial. The following theorem is due to Bernstein [Ber71, Ber72]. It was originally used to define the Bernstein-Sato polynomial. We use a version as in [SST00].

Theorem 4.13 (Bernstein). The Bernstein-Sato polynomial $b_f(s)$ of f is the uniquely determined monic polynomial of minimal degree in $\mathbb{K}[s]$ satisfying the identity

$$P\bullet f^{s+1} = b_f(s)\cdot f^s \qquad \text{for some operator } P\in D_n[s]. \tag{4.1}$$

Proof. Equation (4.1) holds, if and only if

$$0 = P\bullet f^{s+1} - b_f(s)\cdot f^s = (P\cdot f - b_f(s))\bullet f^s,$$

i. e. $P\cdot f - b_f(s)\in \mathrm{Ann}_{D_n[s]}(f^s)$.
In this case, by Theorem 4.12

$$P_{|s=-t\partial_t-1}\cdot f - b_f(-t\partial_t - 1)\in I_f.$$

Since we also have $t-f\in I_f$, it follows that

$$P_{|s=-t\partial_t-1}\cdot f - b_f(-t\partial_t-1) + P_{|s=-t\partial_t-1}(t-f) = -b_f(-t\partial_t-1) + P_{|s=-t\partial_t-1}t \in I_f,$$

which implies

$$\mathrm{in}_{(-w,w)}(b_f(-t\partial_t-1) - P_{|s=-t\partial_t-1}t)\in \mathrm{in}_{(-w,w)}(I_f),$$

for $w=(1,0,\dots,0)\in\mathbb{R}^{n+1}$. Hence, $b_f(-t\partial_t-1)\in \mathrm{in}_{(-w,w)}(I_f)\cap\mathbb{K}[t\partial_t]$, because

$$0 = \deg_{(-w,w)}(b_f(-t\partial_t-1)) = \deg_{(-w,w)}(P_{|s=-t\partial_t-1}) < \deg_{(-w,w)}(P_{|s=-t\partial_t-1}t) = -1.$$

This means, $b_f(s)$ is a multiple of the Bernstein-Sato polynomial.
On the other hand, if $b_f(s)$ is a multiple of the Bernstein-Sato polynomial, then $b_f(-t\partial_t-1)\in \mathrm{in}_{(-w,w)}(I_f)\cap\mathbb{K}[t\partial_t]$, $w=(1,0,\dots,0)\in\mathbb{R}^{n+1}$, by definition. There exists some

$p \in D_n\langle t, \partial_t\rangle$ such that $b_f(-t\partial_t - 1) + p \in I_f$ and $\mathrm{in}_{(-w,w)}(b_f(-t\partial_t - 1) + p) = b_f(-t\partial_t - 1)$. It follows that every monomial of p is divisible by t. Otherwise, $0 = \deg_{(-w,w)}(b_f(-t\partial_t - 1) + p) \leq \deg_{(-w,w)}(p)$, contradicting the latter assumption. By the same argument, every monomial of p that is divisible by ∂_t^k, $k \in \mathbb{N}$, is at least divisible by t^{k+1} as well. Then we can factorize t to the right by Corollary 1.27, i. e. p can be written as

$$\begin{aligned} p &= \left(\sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l} x^\alpha \partial^\beta t^k \partial_t^l\right) t = \left(\sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l} x^\alpha \partial^\beta t^{k-l} t^l \partial_t^l\right) t \\ &= \left(\sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l} x^\alpha \partial^\beta t^{k-l} \left(\prod_{i=0}^{l-1} (t\partial_t - i)\right)\right) t, \end{aligned}$$

where the last equation holds by Corollary 1.26. Since $t - f \in I_f = \mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)$ (Lemma 4.10), computing in $D_n\langle t, \partial_t\rangle / \mathrm{Ann}_{D_n\langle t,\partial_t\rangle}(f^s)$ yields

$$[0] = [b_f(-t\partial_t - 1) + p] = [b_f(-t\partial_t - 1) + \left(\sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l} x^\alpha \partial^\beta f^{k-l} \left(\prod_{i=0}^{l-1} (t\partial_t - i)\right)\right) f].$$

The latter representative is an element of $D_n[t\partial_t]$ and $k - l > 0$ in every non-vanishing term. Applying the algebraic Mellin transform shows that

$$[0] = [b_f(s) + \left(\sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l} x^\alpha \partial^\beta f^{k-l} \cdot \left(\prod_{i=0}^{l-1} (-s - 1 - i)\right)\right) \cdot f]$$

in $D_n[s] / \mathrm{Ann}_{D_n[s]}(f^s)$. Here we have used Theorem 4.12. Setting

$$P := -\sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l} x^\alpha \partial^\beta f^{k-l} \cdot \left(\prod_{i=0}^{l-1} (-s - 1 - i)\right)$$

finally gives us an element of the form $b_f(s) - P \cdot f \in \mathrm{Ann}_{D_n[s]}(f^s)$. $\square$

Now the reason for the substitution of $-s - 1$ by s in the definition of the Bernstein-Sato polynomial (Definition 1.39) is clear: It is nothing else but the algebraic Mellin transform. Recall that the s in the expression $-s - 1$ stands for $t\partial_t$ by definition.

Example 4.14. Consider the following classical example (e. g. [Cou95]).
Let $f := \sum_{i=1}^{n} x_i^2 \in \mathbb{K}[x_1, \ldots, x_n]$ and $P := \sum_{i=1}^{n} \partial_i^2 \in D_n$. Then

$$\begin{aligned} P \bullet f^{s+1} &= \sum_{i=1}^{n} \partial_i^2 \bullet f^{s+1} = \sum_{i=1}^{n} \partial_i \bullet 2x_i(s+1)f^s \\ &= \sum_{i=1}^{n} 2(s+1)f^s + 2x_i s(s+1)2x_i f^{s-1} \\ &= 2n(s+1)f^s + 4s(s+1)f^{s-1} \sum_{i=1}^{n} x_i^2 \\ &= (s+1)(4s+2n)f^s. \end{aligned}$$

Theorem 4.13 implies that $b_f(s) = (s+1)(s+\frac{n}{2})$.

As a consequence of Theorem 4.13 we obtain the claim of Theorem 3.11 for the special case of Bernstein-Sato polynomials.

Lemma 4.15. For non-constant f, the polynomial $s+1$ divides $b_f(s)$ and thus, $b_f(s)$ is non-zero.

Proof. Substituting s with -1 in equation (4.1) yields

$$P(x, \partial, -1) = b_f(-1) \cdot f^{-1}.$$

So we have an element in D_n on the left hand side and a rational function on the right one. Therefore, both expressions are constant. Since f is non-constant, so is f^{-1}. Hence $b_f(-1) = 0$ must hold. □

Moreover, it is known that all roots of the Bernstein-Sato polynomial are negative rational numbers. This fact is due to Kashiwara [Kas76], who gave a proof for so called *local Bernstein-Sato polynomials*, which we do not consider in this work, and the case $\mathbb{K} = \mathbb{C}$. Since the global Bernstein-Sato polynomial is the least common divisor of all local ones, certain flatness properties show that the statement holds [MNM91].

4.3 Bernstein's data

This section is dedicated to the computation of what we like to call *Bernstein's data*, i. e. the triple consisting of the s-parametric annihilator, the Bernstein-Sato polynomial and an operator satisfying Equation (4.1).

Corollary 4.16. In the situation of Theorem 4.13 we have the following statements.

(a) $P \cdot f - b_f(s) \in \mathrm{Ann}_{D_n[s]}(f^s)$ and $b_f(s) \in \mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle$.

(b) $P \notin \operatorname{Ann}_{D_n[s]}(f^{s+1})$.

(c) $\frac{\partial f}{\partial x_i} \cdot P - b'_f(s) \cdot \partial_i \in \operatorname{Ann}_{D_n[s]}(f^{s+1})$, where $b'_f(s) = \frac{1}{s+1} b_f(s)$.

Proof.

(a) The first claim is part of the proof of Theorem 4.13. The second claim follows directly from the first one.

Using equation (4.1) we have

(b) $P \bullet f^{s+1} = b_f(s) \cdot f^s \neq 0$ and

(c) $P \bullet f^{s+1} = b_f(s) \cdot f^s = (s+1) b'_f(s) f^s \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_i}} = (\partial_i \bullet f^{s+1}) b'_f(s) \frac{1}{\frac{\partial f}{\partial x_i}}$. Thus, $\frac{\partial f}{\partial x_i} P \bullet f^{s+1} = b'_f(s) \partial_i \bullet f^{s+1}$ and hence, $\frac{\partial f}{\partial x_i} P - b'_f(s) \partial_i \in \operatorname{Ann}_{D_n[s]}(f^{s+1})$. □

The first part of the corollary provides us with another algorithm to compute the Bernstein-Sato polynomial.

Algorithm 4.17 (`bfctAnn`).
Input: $f \in \mathbb{K}[x_1, \ldots, x_n]$
Output: the Bernstein-Sato polynomial $b_f(s) \in \mathbb{K}[s]$ of f
  $G :=$ a Gröbner basis of $\operatorname{Ann}_{D_n[s]}(f^s) + \langle f \rangle$
  $b_f(s) :=$ `principalIntersect`(s, G) → Algorithm 3.12
  return $b_f(s)$

Proof. By Corollary 4.16(a) and by its definition the Bernstein-Sato polynomial is an element of $\langle G \rangle \cap \mathbb{K}[s]$. Algorithm `principalIntersect` returns the element of minimal degree in this intersection, which is exactly the Bernstein-Sato polynomial by Theorem 4.13. □

Lemma 4.18.

(a) We have $f \partial_i - s \frac{\partial f}{\partial x_i} \in \operatorname{Ann}_{D_n[s]}(f^s)$ for all $1 \leq i \leq n$.

(b) $\left(\operatorname{Ann}_{D_n[s]}(f^s) + \langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle \right) \cap \mathbb{K}[s] = \langle \frac{b_f(s)}{s+1} \rangle$.

Proof. Let us abbreviate $f_i := \frac{\partial f}{\partial x_i}$, $1 \leq i \leq n$.

(a) $(f \partial_i - s f_i) \bullet f^s = f \partial_i \bullet f^s - s f_i \bullet f^s = f s f^{s-1} f_i - s f_i f^s = 0$.

(b) Let $P \in D_n[s]$ be an operator satisfying Equation (4.1). Let us write P in the form $P = P_0 + \sum_{i=1}^n P_i \partial_i$ such that $P_0 \in \mathbb{K}[x, s]$ and $P_i \in D_n[s]$, $1 \leq i \leq n$. Computing modulo $\mathrm{Ann}_{D_n[s]}(f^s)$ and using Corollary 1.25 and (a) we have

$$\begin{aligned} P \cdot f &= P_0 f + \sum_{i=1}^{n} P_i \partial_i f = P_0 f + \sum_{i=1}^{n} P_i (f \partial_i + f_i) \\ &= P_0 f + \sum_{i=1}^{n} P_i (s f_i + f_i) = P_0 f + (s+1) \sum_{i=1}^{n} P_i f_i. \end{aligned}$$

By Lemma 4.15 $b_f(-1) = 0$, hence by restricting s to -1 in Equation (4.1) we get $P_{|s=-1} \bullet 1 = b_f(-1) f^{-1} = 0$. Thus, $P_{|s=-1} \in \mathrm{Ann}_{D_n[s]}(1) = \langle \partial_1, \ldots, \partial_n \rangle$. On the other hand, $P_{|s=-1} f = P_{0|s=-1} f$. But then $P_{0|s=-1} = P_{|s=-1} \in \langle \partial_1, \ldots, \partial_n \rangle$, which means $P_{0|s=-1} = 0$ by the choice of P_0. Hence, $s+1$ divides P_0. Moreover, by Corollary 4.16(a)

$$0 = (Pf - b_f) \bullet f^s = ((s+1)(\frac{P_0}{s+1} f + \sum_{i=1}^{n} P_i f_i - \frac{b_f}{s+1})) \bullet f^s.$$

Since $s+1 \notin \mathrm{Ann}_{D_n[s]}(f^s)$, it follows that

$$\frac{P_0}{s+1} f + \sum_{i=1}^{n} P_i f_i - \frac{b_f}{s+1} \in \mathrm{Ann}_{D_n[s]}(f^s).$$

Thus,

$$\langle \frac{b_f(s)}{s+1} \rangle = \big(\mathrm{Ann}_{D_n[s]}(f^s) + \langle f, f_1, \ldots, f_n \rangle \big) \cap \mathbb{K}[s]. \qquad \square$$

When computing the Bernstein-Sato polynomial with Algorithm 4.17, adding all partial derivatives of f to $\mathrm{Ann}_{D_n}(f) + \langle f \rangle$ will improve the efficiency of the algorithm since we have to consider one normal form less with `pIntersect`. See also [ALMM10].

4.3.1 Computing s-parametric annihilators

As one can see, `bfctAnn` requires the computation of $\mathrm{Ann}_{D_n[s]}(f^s)$. There are several approaches known, see for example [LMM08]. Here, we will only give the idea behind the algorithm by Briançon and Maisonobe [BM02], which seems to be the most efficient one in practice.

Theorem 4.19 (Briançon-Maisonobe, 2002). Consider the *shift algebra* $S := \mathbb{K}\langle \partial_t, \sigma \mid \sigma \partial_t = \partial_t \sigma + \partial_t \rangle$ and $D_n^S := D_n \otimes_{\mathbb{K}} S$. Further, for $f \in \mathbb{K}[x_1, \ldots, x_n]$ define

$$I := \langle \sigma + f \cdot \partial_t, \{\partial_i + \frac{\partial f}{\partial x_i} \cdot \partial_t\} \rangle \subseteq D_n^S.$$

Then $\mathrm{Ann}_{D_n[s]}(f^s) = I_{|\sigma=s} \cap D_n[s]$.

Note that the relation $\sigma\partial_t = \partial_t\sigma + \partial_t$ in S corresponds to the relation $(-t\partial_t)\partial_t = \partial_t(-t\partial_t) + \partial_t$ in $D_n\langle t, \partial_t\rangle$, i. e. D_n^S is isomorphic to $D_n\langle \partial_t, t\partial_t\rangle \subseteq D_n\langle t, \partial_t\rangle$.

The theorem directly gives rise to the following algorithm, implemented in the SINGULAR library `dmod.lib` [LMM10].

Algorithm 4.20 (`SannfsBM`).
Input: $f \in \mathbb{K}[x_1, \dots, x_n]$
Output: $\mathrm{Ann}_{D_n[s]}(f^s) \subseteq D_n[s]$
$I := \langle \sigma + f \cdot \partial_t, \{\partial_i + \frac{\partial f}{\partial x_i} \cdot \partial_t\}\rangle \subseteq D_n^S$
$G :=$ a Gröbner basis of I with respect to an elimination ordering for ∂_t
return $G_{|\sigma=s} \cap D_n[s]$

A purely computer algebraic proof of Theorem 4.19 respectively Algorithm 4.20 can be found in [ALMM09].

Our next goal is to improve the previous algorithm by obtaining a pre-processing for the required elimination of ∂_t.

Remark 4.21. Consider the *Jacobian matrix* J_f of f, that is the matrix of all partial derivatives of f,

$$J_f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \in \mathbb{K}[x_1, \dots, x_n]^{1\times n}.$$

Let $J = (f, J_f) \in \mathbb{K}[x_1, \dots, x_n]^{1\times(n+1)}$ and $y = (y_f, y_1, \dots, y_n)^{tr} \in \mathbb{K}[x_1, \dots, x_n]^{(n+1)\times 1}$ such that $y^{tr} \cdot J^{tr} = 0$ holds, i. e. y is a *(left) syzygy* of J. In Algorithm 4.20 we have to consider the ideal generated by the entries of the matrix

$$I := \left(\sigma + f \cdot \partial_t, \partial_1 + \frac{\partial f}{\partial x_1} \cdot \partial_t, \dots, \partial_n + \frac{\partial f}{\partial x_n} \cdot \partial_t\right) \in (D_n^S)^{1\times(n+1)}.$$

Multiplying y with I results in an element of D_n^S of the form

$$\begin{aligned} y^{tr} \cdot I^{tr} &= y_f \cdot (\sigma + f \cdot \partial_t) + \sum_{i=1}^n y_i \cdot (\partial_i + \frac{\partial f}{\partial x_i} \cdot \partial_t) \\ &= y_f \cdot \sigma + (y_f \cdot f + \sum_{i=1}^n y_i \cdot \frac{\partial f}{\partial x_i}) \cdot \partial_t + \sum_{i=1}^n y_i \cdot \partial_i \\ &= y_f \cdot \sigma + (y^{tr} \cdot J^{tr}) \cdot \partial_t + \sum_{i=1}^n y_i \cdot \partial_i \\ &= y_f \cdot \sigma + \sum_{i=1}^n y_i \cdot \partial_i. \end{aligned}$$

Examining the action of this operator on f^s under the substitution of σ with s yields

$$\begin{aligned}
(y^{tr} \cdot I^{tr})_{|\sigma=s} \bullet f^s &= y_f \cdot s \bullet f^s + \sum_{i=1}^n y_i \cdot \partial_i \bullet f^s \\
&= y_f \cdot s \cdot f^s + \sum_{i=1}^n y_i \cdot s \cdot \frac{\partial f}{\partial x_i} \cdot f^{s-1} \\
&= (y_f \cdot f + \sum_{i=1}^n y_i \cdot \frac{\partial f}{\partial x_i}) \cdot s \cdot f^{s-1} \\
&= (y^{tr} \cdot J^{tr}) \cdot s \cdot f^{s-1} = 0.
\end{aligned}$$

Hence, every (left) syzygy y of J induces an element of $\operatorname{Ann}_{D_n[s]}(f^s)$, namely $y^{tr} \cdot I^{tr}$, which also has the property of being linear in $\partial_1, \ldots, \partial_n$, i. e. the differential variables only appear in degree at most one. The ideal generated by those elements is called the *logarithmic annihilator* of f^s.

Now, we can slightly modify Algorithm 4.20 as follows.

Algorithm 4.22 (`SannfsBMSyz`).
Input: $f \in \mathbb{K}[x_1, \ldots, x_n]$
Output: $\operatorname{Ann}_{D_n[s]}(f^s) \subseteq D_n[s]$
$J_f :=$ the Jacobian matrix of f
$M :=$ the first syzygy module of $(f, J_f) \in \mathbb{K}[x_1, \ldots, x_n]^{1\times(n+1)}$
$G_M :=$ a Gröbner basis of M
$B := \left(\sigma + f \cdot \partial_t, \partial_1 + \frac{\partial f}{\partial x_1} \cdot \partial_t, \ldots, \partial_n + \frac{\partial f}{\partial x_n} \cdot \partial_t\right) \in (D_n^S)^{1\times(n+1)}$
$A := \{(y^{tr} \cdot B^{tr})_{|\sigma=s} \mid y \in G_M\} \subseteq D_n[s]$
$G_A :=$ a Gröbner basis of $\langle A \rangle$
$I :=$ the ideal generated by the entries of B
$G :=$ a Gröbner basis of $G_{A|_{s=\sigma}} + I$ with respect to an elimination ordering for ∂_t
return $\langle G_{|\sigma=s} \cap D_n[s] \rangle$

The advantage of this modification is that we already start the elimination computation with a partial answer, which also consists of elements of small degree, since ∂_i only appears in the elements of A in degree at most one as stated above. See Section 6.2.4 for experimental results.

4.3.2 Computing b-operators

We now study the operators satisfying Equation (4.1).

Remark 4.23. Let $P, Q \in D_n[s]$ be two operators satisfying Equation (4.1). Then $P \bullet f^{s+1} = Q \bullet f^{s+1}$, i. e. $(P - Q) \bullet f^{s+1} = 0$, which means $P - Q \in \operatorname{Ann}_{D_n[s]}(f^{s+1})$. Hence, such an operator is uniquely determined up to $\operatorname{Ann}_{D_n[s]}(f^{s+1})$.

Definition 4.24. We call an operator $P \in D_n[s]$ satisfying Equation (4.1) a *b-operator*. Let P be a b-operator. Then we call the b-operator $\mathrm{NF}(P, \mathrm{Ann}_{D_n[s]}(f^{s+1}))$, the *Bernstein operator*.

Note that the Bernstein operator is uniquely determined because the normal form is. We present four distinguished methods to compute a b-operator.

Linear dependency

Our first approach to compute a b-operator is based on Corollary 4.16(a) and on the idea of Algorithm 3.12: We search for a linear dependency between $b_f(s)$ and $\{m \cdot f \mid m \in D_n[s] \text{ monomial}\}$ in $D_n[s]/\operatorname{Ann}_{D_n[s]}(f^s)$.

Algorithm 4.25 (`bOperatorLinDep`).

Input: $f \in \mathbb{K}[x_1, \ldots, x_n]$, the Bernstein-Sato polynomial $b_f(s)$ of f, $\mathrm{Ann}_{D_n[s]}(f^s)$
Output: $P \in D_n[s]$ such that $P \bullet f^{s+1} = b_f(s) \cdot f^s$
  $d := 0$
  loop
    $M_d := \{m \in D_n[s] \setminus \mathrm{Ann}_{D_n[s]}(f^{s+1}) \mid m \text{ monomial}, \deg(m) \leq d\}$
    if there exist $a_m \in \mathbb{K}$ such that $b_f(s) = \sum_{m \in M_d} a_m \, \mathrm{NF}(m \cdot f, \mathrm{Ann}_{D_n[s]}(f^s))$ **then**
      return $P := \sum_{m \in M_d} a_m m - b_f(s)$
    else
      $d := d + 1$
    end if
  end loop

Lifting

Remark 4.26. Let $I \subseteq J$ be two ideals in a G-algebra A and let I and J be generated by the sets $F := \{f_1, \ldots, f_m\}$ and $G := \{g_1, \ldots, g_l\}$, respectively. Then each $f_i \in F$ has a representation in terms of G, i. e. there exists a matrix $T \in A^{l \times m}$ such that $[f_1, \ldots, f_m]^{tr} = T^{tr} \cdot [g_1, \ldots, g_l]^{tr}$. We call T a *lifting matrix*. Moreover, T can be computed with the Singular command `lift`(G, F).

Algorithm 4.27 (`bOperatorLift`).

Input: $f \in \mathbb{K}[x_1, \ldots, x_n]$, the Bernstein-Sato polynomial $b_f(s)$ of f, $\mathrm{Ann}_{D_n[s]}(f^s)$
Output: $P \in D_n[s]$ such that $P \bullet f^{s+1} = b_f(s) \cdot f^s$
  $\{h_1, \ldots, h_m\}$ generators of $\mathrm{Ann}_{D_n[s]}(f^s)$
  $T :=$ `lift`$([f, h_1, \ldots, h_m], [b_f(s)])$
  return $T_{1,1}$

Proof. Since $b_f(s) \in \langle f\rangle + \operatorname{Ann}_{D_n[s]}(f^s)$, `lift` is applicable and yields a representation

$$b_f(s) = T_{1,1}f + \sum_{i=2}^{m} T_{i,1}h_i.$$

Since $\sum_{i=2}^{m} T_{i,1}h_i \in \operatorname{Ann}_{D_n[s]}(f^s)$, Corollary 4.16(a) proves the correctness of the algorithm. □

Kernel of a module homomorphism

Let e_i denote the i-th standard basis vector and consider the homomorphism of $D_n[s]$-modules

$$\phi : D_n[s]^2 \to D_n[s]/\operatorname{Ann}_{D_n[s]}(f^s), e_1 \mapsto b_f(s), e_2 \mapsto f.$$

Then $\ker(\phi) = \{(u,v)^{tr} \in D_n[s]^2 \mid ub_f(s) + vf \in \operatorname{Ann}_{D_n[s]}(f^s)\}$. Any $(u,v)^{tr} \in \ker(\phi)$ with $0 \neq u \in \mathbb{K}$ induces a b-operator, namely $u^{-1}v$. Since the existence of b-operators is guaranteed, so is the existence of such an element. Moreover, it can be computed via Gröbner bases for modules with respect to a *position over term ordering* preferring the first component. If such a Gröbner basis is reduced, it contains exactly one element with this property. In addition, $\ker(\phi)$ can be computed with the Singular commands `modulo` or `moduloSlim`, respectively.

Kernel of a module homomorphism reloaded

Note that the second part of the proof of Theorem 4.13 is constructive. That is, it provides a way to compute a b-operator. We get the following algorithm.

Algorithm 4.28 (`bOperatorModulo`).
Input: $f \in \mathbb{K}[x_1,\dots,x_n]$, the Bernstein-Sato polynomial $b_f(s) \in \mathbb{K}[s]$ of f
Output: a b-operator $P \in D_n[s]$, i. e. $P \bullet f^{s+1} = b_f(s)\cdot f^s$
  $I_f \subseteq D_n\langle t,\partial_t\rangle$ the Malgrange ideal of f
  $w := (1,0,\dots,0) \in \mathbb{R}^{n+1}$
  $p \in D_n\langle t,\partial_t\rangle$ such that $b_f(-t\partial_t - 1) + p\cdot t \in I_f$ and $\deg_{(-w,w)}(p) \leq 0$
  find a representation of p of the form $p = \sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l}x^\alpha\partial^\beta t^{k-l}\left(\prod_{i=0}^{l-1}(t\partial_t - i)\right)$
  return $-\sum_{\alpha,\beta,k,l} c_{\alpha,\beta,k,l}x^\alpha\partial^\beta f^{k-l}\left(\prod_{i=0}^{l-1}(-s-1-i)\right)$

Note that p from the previous algorithm can be computed using the kernel of the homomorphism of $D_n\langle t,\partial_t\rangle$-modules

$$\psi : D_n\langle t,\partial_t\rangle^2 \to D_n\langle t,\partial_t\rangle/I_f, e_1 \mapsto b_f(-t\partial_t - 1), e_2 \mapsto t$$

analogously to the previous method.

5 Applications of b-functions

We briefly describe some of the many applications of b-functions.

5.1 Annihilators of powers of polynomials

In the previous chapter, we have investigated the s-parametric annihilator. Now we will turn our attention to the annihilator of (non-symbolic) powers of polynomials.

Example 5.1. Consider the univariate polynomial $f := x \in \mathbb{K}[x]$. We compute $\operatorname{Ann}_{D_1[s]}(f^s)$. According to Algorithm 4.20, $\operatorname{Ann}_{D_1[s]}(x^s) = I_{|\sigma=s} \cap D_1[s]$, where $I = \langle \sigma + x\cdot\partial_t, \partial_x + \partial_t \rangle \subseteq D_1 \otimes_{\mathbb{K}} \mathbb{K}\langle \partial_t, \sigma \mid \sigma\partial_t = \partial_t\sigma + \partial_t \rangle$. Apparently, $\operatorname{Ann}_{D_1[s]}(x^s) = \langle x\partial_x - s \rangle$. Now, we would like to compute $\operatorname{Ann}_{D_1}(x^\lambda)$ for a specific $\lambda \in \mathbb{C}$. Since $(x\partial_x - \lambda) \bullet x^\lambda = x\lambda x^{\lambda-1} - \lambda x^\lambda = 0$, we have $\operatorname{Ann}_{D_1[s]}(x^s)_{|s=\lambda} \subseteq \operatorname{Ann}_{D_1}(x^\lambda)$ as expected. But in the case $\lambda \in \mathbb{N}$, restricting the s-parametric annihilator does not yield the full annihilator of x^λ because also $\partial_x^{\lambda+1} \bullet x^\lambda = 0$.

The previous example raises a few questions. When is it sufficient to just restrict the s-parametric annihilator in order to compute $\operatorname{Ann}_{D_n}(f^\lambda)$? What can we do in the cases where it does not suffice?

Lemma 5.2. Let $f \in \mathbb{K}[x_1, \dots, x_n] \setminus \mathbb{K}$. Then f^{s-i}, $i \in \mathbb{N}_0$, can be written in the form

$$f^{s-i} = \frac{P_i}{\prod_{j=1}^{i} b_f(s-j)} \bullet f^s \qquad \text{for some } P_i \in D_n[s].$$

Proof. We prove the lemma by induction on i. For $i = 0$ there is nothing to do by the convention that the empty product equals 1. Recall from Theorem 4.13 that there exists a b-operator $P \in D_n[s]$, i. e. P satisfies $P \bullet f^{s+1} = b_f(s) \cdot f^s$. Substituting s with $s-1$ in this equation yields $P_{|s=s-1} \bullet f^s = b_f(s-1) \cdot f^{s-1}$ or equivalently $\frac{P_{|s=s-1}}{b_f(s-1)} \bullet f^s = f^{s-1}$ by the action of the operation $\bullet$ on s, which shows the case $i = 1$. Now assume the claim holds for some $i \in \mathbb{N}$. Substituting s with $s-i-1$ in the defining equation of the b-operator gives $P_{|s=s-i-1} \bullet f^{s-i} = b_f(s-i-1) \cdot f^{s-i-1}$. By rewriting the latter equation and using the induction hypothesis, we have

$$\begin{aligned} f^{s-i-1} &= \frac{P_{|s=s-i-1}}{b_f(s-i-1)} \bullet f^{s-i} = \frac{P_{|s=s-i-1}}{b_f(s-i-1)} \bullet \left(\frac{P_i}{\prod_{j=1}^{i} b_f(s-j)} \bullet f^s \right) \\ &= \frac{P_{|s=s-i-1} \cdot P_i}{\prod_{j=1}^{i+1} b_f(s-j)} \bullet f^s. \end{aligned}$$

□

Here is where the Bernstein-Sato polynomial comes into play.

Theorem 5.3. Let $f \in \mathbb{K}[x_1, \ldots, x_n] \setminus \mathbb{K}$ and $\lambda_0 := \min\{\lambda \in \mathbb{Z} \mid b_f(\lambda) = 0\}$ be the minimal integral root of the Bernstein-Sato polynomial b_f of f. Further, let $\lambda \in \mathbb{K} \setminus \{\lambda_0 + k \mid k \in \mathbb{N}\}$. Then $\operatorname{Ann}_{D_n}(f^\lambda) = \operatorname{Ann}_{D_n[s]}(f^s)_{|s=\lambda}$.

Proof. Clearly, the inclusion $\operatorname{Ann}_{D_n[s]}(f^s)_{|s=\lambda} \subseteq \operatorname{Ann}_{D_n}(f^\lambda)$ holds.
Let $P \in \operatorname{Ann}_{D_n}(f^\lambda)$. We construct an operator $Q \in \operatorname{Ann}_{D_n[s]}(f^s)$ such that $Q_{|s=\lambda} = P$. Let $r := \deg_{(0,\ldots,0,1,\ldots,1)}(P)$ denote the total degree in ∂ of P. By the action of ∂_j on f^s, $P \bullet f^s$ takes then the form $P \bullet f^s = \sum_{i=0}^{r} g_i' \cdot f^{s-i}$ for $g_i' \in \mathbb{K}[x, s]$, $1 \le i \le r$. Without loss of generality assume that f does not divide any g_i'. Since P does not contain s, we have

$$0 = P \bullet f^\lambda = P \bullet f^s{}_{|s=\lambda} = \sum_{i=0}^{r} (g_i' \cdot f^{s-i})_{|s=\lambda} = \sum_{i=0}^{r} g'_{i|s=\lambda} \cdot f^{\lambda-i},$$

which shows that $g'_{i|s=\lambda} = 0$ for all i. Therefore we can write $P \bullet f^s = (s-\lambda) \sum_{i=0}^{r} g_i \cdot f^{s-i}$ with $g_i \in \mathbb{K}[x, s]$, $1 \le i \le r$. By Lemma 5.2, there exist $P_i \in D_n[s]$ such that

$$P \bullet f^s = (s-\lambda) \sum_{i=0}^{r} g_i \cdot \frac{P_i}{\prod_{j=1}^{i} b_f(s-j)} \bullet f^s.$$

Hence,

$$P - (s-\lambda) \sum_{i=0}^{r} g_i \cdot \frac{P_i}{\prod_{j=1}^{i} b_f(s-j)} \in \operatorname{Ann}_{D_n[s]}(f^s),$$

which implies that

$$Q' := \left(\prod_{j=1}^{r} b_f(s-j) \right) \cdot P - (s-\lambda) \sum_{i=0}^{r} g_i \cdot P_i \cdot \prod_{j=i+1}^{r} b_f(s-j) \in \operatorname{Ann}_{D_n[s]}(f^s).$$

Moreover, $\prod_{j=1}^{r} b_f(\lambda-j) \neq 0$ since $\lambda \notin \{\lambda_0 + k \mid k \in \mathbb{N}\}$. By setting $Q := \frac{1}{\prod_{j=1}^{r} b_f(\lambda-j)} \cdot Q'$ we obtain an operator as desired. □

Now we can formulate an algorithm to compute the annihilator of f^λ for any $\lambda \in \mathbb{C}$.

Algorithm 5.4 (`annflambda`).
Input: $f \in \mathbb{K}[x_1, \ldots, x_n], \lambda \in \mathbb{C}$
Output: $\mathrm{Ann}_{D_n}(f^\lambda) \subseteq D_n$
$G := \{g_1, \ldots, g_r\}$ a Gröbner basis of $\mathrm{Ann}_{D_n[s]}(f^s) \subseteq D_n[s]$ → Algorithm 4.20
$\lambda_0 := \min\{\lambda \in \mathbb{Z} \mid b_f(\lambda) = 0\}$ → Algorithm 4.17
$d := \lambda - \lambda_0, \quad G' := \emptyset$
if $d \in \mathbb{N}$ **then**
$M := \{(c_0, c_1, \ldots, c_r) \in D_n^{r+1} \mid c_0 f^d + \sum_{i=1}^r c_i g_{i_{|s=\lambda_0}} = 0\} \subseteq D_n^{r+1}$ the first syzygy module of $(f^d, g_{1_{|s=\lambda_0}}, \ldots, g_{r_{|s=\lambda_0}}) \in D_n^{r+1}$
$G' = \{c_0 \mid (c_0, c_1, \ldots, c_r) \in M\} \subseteq D_n$
end if
return $G_{|s=\lambda} \cup G'$

Proof. If $d = \lambda - \lambda_0 \notin \mathbb{N}$, then $\mathrm{Ann}_{D_n}(f^\lambda) = \mathrm{Ann}_{D_n[s]}(f^s)_{|s=\lambda}$ by Theorem 5.3. Otherwise, consider $c_0 \in D_n$. We have

$$c_0 \bullet f^\lambda = c_0 \bullet (f^d f^{\lambda_0}) = (c_0 \cdot f^d) \bullet f^{\lambda_0},$$

i. e. $c_0 \in \mathrm{Ann}_{D_n}(f^\lambda)$ if and only if $c_0 f^d \in \mathrm{Ann}_{D_n}(f^{\lambda_0}) = \mathrm{Ann}_{D_n[s]}(f^s)_{|s=\lambda_0}$ again by Theorem 5.3. Since $\mathrm{Ann}_{D_n[s]}(f^s) = \langle g_1, \ldots, g_r \rangle$, this is the case if and only if there exist $c_1, \ldots, c_r \in D_n$ such that $c_0 f^d + \sum_{i=1}^r c_i g_{i_{|s=\lambda_0}} = 0$, which shows the correctness of the algorithm. $\square$

Example 5.5. Let $f := x^2 + ax + b \in \mathbb{Q}[x, a, b]$. We compute $\mathrm{Ann}_{D_3}(f^{-1})$ and $\mathrm{Ann}_{D_3}(f)$.

```
LIB "bfun.lib";
ring r = 0,(x,a,b),dp;
poly f = x^2+a*x+b;
bfct(f);
==> [1]:
==>    _[1]=-1
==> [2]:
==>    1
```

So $b_f(s) = s + 1$ and hence, its minimal integral root is -1. The procedure `SannfsBM` below (see Algorithm 4.20) returns $D_3[s]$, which contains an object of the type `ideal` called `LD` being $\mathrm{Ann}_{D_3[s]}(f^s)$.

```
def D3s = SannfsBM(f); setring D3s; LD;
==> LD[1]=a*Db-Dx+2*Da
==> LD[2]=x*Db-Da
==> LD[3]=x*Dx-2*x*Da-a*Da
==> LD[4]=x*Da+a*Da+b*Db-s
```

Since $-1-(-1)=0\notin\mathbb{N}$ we get $I := \mathrm{Ann}_{D_3}(f^{-1})$ by substitution:

```
ideal I = std(subst(LD,s,-1)); I;
==> I[1]=a*Db-Dx+2*Da
==> I[2]=x*Db-Da
==> I[3]=x*Da+a*Da+b*Db+1
==> I[4]=x*Dx-2*x*Da-a*Da
==> I[5]=b*Db^2+Dx*Da-Da^2+Db
==> I[6]=a*Dx*Da+2*x*Da^2+a*Da^2+b*Dx*Db+Dx+2*Da
```

So I is a Gröbner basis of $\mathrm{Ann}_{D_3}(f^{-1})$. Let us continue with the computation of $J := \mathrm{Ann}_{D_3}(f)$. Since $1-(-1)=2\in\mathbb{N}$ we need to compute syzygies.

```
ring r2 = 0,(x,a,b,Dx,Da,Db),dp;
def D3 = Weyl(); setring D3;   // create D3
poly f = imap(r,f);            // fetch f from r to current ring D3
ideal I = imap(D3s,I);         // fetch I from D3s to current ring D3
matrix M = matrix(syz(f^2+I)); // first syzygy module given as matrix
ideal MM = (M[1,1..ncols(M)]); // first components of the generators
```

Now we need to add $I_2 := \mathrm{Ann}_{D_n[s]}(f^s)_{|_{s=1}}$ to get J:

```
map m = D3s,maxideal(1),1;     // define homomorphism D3[s] → D3,
ideal I2 = m(LD);              // that sends s to 1
ideal J = std(MM+I2); J;       // sum of ideals, cosmetic Gröbner basis
==> J[1]=Db^2
==> J[2]=Da*Db
==> J[3]=Dx*Db
==> J[4]=a*Db-Dx+2*Da
==> J[5]=x*Db-Da
==> J[6]=Da^2
==> J[7]=Dx*Da-2*Da^2-Db
==> J[8]=x*Da+a*Da+b*Db-1
==> J[9]=Dx^2-2*Db
==> J[10]=x*Dx+a*Da+2*b*Db-2
```

Hence, J is a Gröbner basis of $\mathrm{Ann}_{D_3}(f)$.

5.2 Restriction

Let $f(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$ be a function (not necessarily a polynomial one). Suppose we know a holonomic ideal $I \subseteq D_n$ such that

$$I \subseteq \operatorname{Ann}_{D_n}(f(x_1, \dots, x_m, x_{m+1}, \dots, x_n)).$$

Let D_{n-m} denote the $(n-m)$-th Weyl algebra

$$\mathbb{K}\langle x_{m+1}, \dots, x_n, \partial_{m+1}, \dots, \partial_n \mid \{\partial_i x_j = x_j \partial_i + \delta_{ij} \mid m+1 \leq i, j \leq n\}\rangle.$$

Our goal is to compute an ideal $J \subseteq D_{n-m}$ such that

$$J \subseteq \operatorname{Ann}_{D_{n-m}}(f(0, \dots, 0, x_{m+1}, \dots, x_n))$$

directly from I. In particular, we do not require to know f directly. Note that this corresponds to the usual setup in algebraic analysis, where a function is given in terms of its annihilator and a finite number of initial values.

Remark 5.6. Consider the right ideal $\langle x_1, \dots, x_m \rangle_{D_n} \subseteq D_n$. The quotient of $\mathbb{K}$-vector spaces $D_n / \langle x_1, \dots, x_m \rangle_{D_n}$ is a right D_n-module and also a left D_{n-m}-module. Thus, it can be viewed as a D_{n-m}-D_n-bimodule. Any left ideal $I \subseteq D_n$ can be viewed as a D_n-$\mathbb{K}$-bimodule. Thus,

$$D_n / \langle x_1, \dots, x_m \rangle_{D_n} \otimes_{D_n} D_n / I \cong D_n / (I + \langle x_1, \dots, x_m \rangle_{D_n}) =: R_m$$

has the structure of a D_{n-m}-$\mathbb{K}$-bimodule and in particular, of a left D_{n-m}-module. We refer to [Cou95] for the details. The left D_{n-m}-module R_m is called the *restriction module* of D_n/I or simply the *restriction* of D_n/I with respect to $x_1, \dots, x_m$.

Theorem 5.7. Let $f(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$ be a given function and

$$I \subseteq \operatorname{Ann}_{D_n}(f(x_1, \dots, x_m, x_{m+1}, \dots, x_n))$$

a holonomic ideal in D_n. Then

$$(I + \langle x_1, \dots, x_m \rangle_{D_n}) \cap D_{n-m} \subseteq \operatorname{Ann}_{D_{n-m}}(f(0, \dots, 0, x_{m+1}, \dots, x_n)).$$

Proof. Any $p \in (I + \langle x_1, \dots, x_m \rangle_{D_n}) \cap D_{n-m}$ can be written in the form

$$p = q + \sum_{i=1}^{n} x_i r_i \qquad \text{with } q \in I \text{ and suitable } r_i \in D_n,$$

which implies that

$$p \bullet f = q \bullet f + \left(\sum_{i=1}^{m} x_i r_i \right) \bullet f = \sum_{i=1}^{m} x_i (r_i \bullet f).$$

Since p does not contain any $x_1, \ldots, x_m$, we have

$$p \bullet f(0, \ldots, 0, x_{m+1}, \ldots, x_m) = p \bullet f_{|x_1=0,\ldots,x_m=0} = \left(\sum_{i=1}^{m} x_i (r_i \bullet f) \right)_{|x_1=0,\ldots,x_m=0} = 0. \quad \square$$

Note that $D_{n-m}/((I + \langle x_1, \ldots, x_m \rangle_{D_n}) \cap D_{n-m})$ is the cyclic submodule of R_m generated by 1. The idea of what follows now is that certain roots of a suitable b-function give bounds.

Theorem 5.8 ([SST00, Theorem 5.2.6]). Let $w \in \mathbb{R}^n$ be a weight vector satisfying $w_i > 0$ for $1 \leq i \leq m$ and $w_i = 0$ for $m+1 \leq i \leq n$ and let $\lambda_0 \in \mathbb{Z}$ be an integer with the property $\lambda_0 \geq \max\{\alpha \in \mathbb{Z} \mid b_{I,w}(\alpha) = 0\}$. Further, let $I \subseteq D_n$ be a holonomic ideal and $G = \{g_1, \ldots, g_r\} \subseteq D_n$ such that $\mathrm{in}_{(-w,w)}(G)$ is a Gröbner basis of $\mathrm{in}_{(-w,w)}(I)$ (cf. Theorems 2.29, 2.31) and put $m_i := \deg_{(-w,w)}(g_i)$ for $1 \leq i \leq r$. Consider the V-filtration with respect to $(-w, w)$ defined by $V_k = \{\sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta \mid -w\alpha + w\beta \leq k\} \subseteq D_n$ for $k \in \mathbb{Z}$ (see Example 2.4). Then

$$R_m = D_n / (I + \langle x_1, \ldots, x_m \rangle_{D_n}) \cong V_{\lambda_0} \bigg/ \left(\sum_{i=1}^{r} V_{\lambda_0 - m_i} g_i + \sum_{j=1}^{m} x_j V_{\lambda_0 + w_j} \right)$$

as left D_{n-m}-modules.

A Gröbner basis G such that $\mathrm{in}_{(-w,w)}(G)$ is a Gröbner basis of $\mathrm{in}_{(-w,w)}(I)$ can be computed by a slight modification of Algorithm 2.32. That is instead of returning $\mathrm{in}_{(u,v)}(G)$ in the last line of Algorithm 2.32 one can return G to get the desired property.

Corollary 5.9. If $w \in \mathbb{R}^n$ is a weight satisfying the conditions in the previous theorem and $b_{I,w}$ has no non-negative integral root, then the restriction module R_m equals $\{0\}$.

Proof. With notations as in the previous theorem, we can put $\lambda_0 = -1$ since all integral roots of $b_{I,w}$ are non-negative. Further, every monomial of strictly negative weighted total degree with respect to $(-w, w)$ contains one of the variables $x_1, \ldots, x_m$. Hence, the equality $V_{-1} = \sum_{j=1}^{m} x_j V_0$ holds. Moreover, since $V_i \subseteq V_j$ for $i \leq j$ and $w_j > 0$ for $1 \leq j \leq m$, we have $V_{-1} \subseteq V_{w_j - 1}$ for $1 \leq j \leq m$, which implies

$$V_{-1} \bigg/ \left(\sum_{i=1}^{r} V_{-(m_i+1)} g_i + \sum_{j=1}^{m} x_j V_{w_j - 1} \right) = \{0\}.$$

Thus, $R_m = \{0\}$ as well by the isomorphism above. $\square$

Note that the existence of an integral root of $b_{I,w}$ is not guaranteed, in contrast to b-functions of polynomials, which always have the root -1 according to Lemma 4.15. Further note that the restriction module R_m is *finitely presented*, i. e. R_m is isomorphic to D^k_{n-m}/M for a submodule $M \subseteq D^k_{n-m}$ and some $k \in \mathbb{N}$, see e. g. [Cou95].

Algorithm 5.10 (`restrictionModule`).

Input: $I \subseteq D_n$ holonomic
Output: a presentation D_{n-m}^k/M of the restriction module R_m

$w \in \mathbb{R}^n$ such that $w_i > 0$ for $1 \le i \le m$ and $w_i = 0$ for $m+1 \le i \le n$
$\lambda_0 := \max\{-1, \lambda \mid \lambda \in \mathbb{Z}, b_f(\lambda) = 0\}$ $\rightarrow$ Algorithm 3.16
if $\lambda_0 < 0$ **then**
  return $R_m = \{0\}$
end if
$G = \{g_1, \ldots, g_r\}$ a Gröbner basis of I with respect to a global ordering such that
  $\mathrm{in}_{(-w,w)}(G)$ is a Gröbner basis of $\mathrm{in}_{(-w,w)}(I)$ $\rightarrow$ analogous to Algorithm 2.32
$m_i := \deg_{(-w,w)}(g_i)$ for $1 \le i \le r$, $M' := \emptyset$
$B_l := \{\partial_1^{\beta_1} \cdots \partial_m^{\beta_m} \mid \sum_{j=1}^m w_j\beta_j \le l\}$ for $l \in \{\lambda_0, \lambda_0 - m_i \mid 1 \le i \le r\}$
for $1 \le i \le r$ **do**
  $M' := M' \cup \{\left(\partial^\beta \cdot g_i\right)_{|x_1=0,\ldots,x_m=0} \mid \partial^\beta \in B_{\lambda_0 - m_i}\}$
end for
$M :=$ the D_{n-m}-submodule of $D_{n-m} \cdot B_{\lambda_0}$ generated by M'
return $(D_{n-m} \cdot B_{\lambda_0})/M$

Proof. The sets of monomials B_{λ_0} and $B_{\lambda_0 - m_i}, 1 \le i \le r$, are finite because $w_i > 0$ for $1 \le i \le r$. The choice of G ensures the compatibility with the V-filtration (cf. the proof of Theorem 2.31). In particular, $\left(\partial^\beta \cdot g_i\right)_{|x_1=0,\ldots,x_m=0} \in B_{\lambda_0}$ for $\partial^\beta \in B_{\lambda_0 - m_i}$, $1 \le i \le r$. For any $k \in \mathbb{Z}$, we have $\sum_{i=1}^m x_i \cdot V_{k+w_i} = (\sum_{i=1}^m x_i \cdot D_n) \cap V_k = \langle x_1, \ldots, x_m \rangle_{D_n} \cap V_k$ and $V_k / \sum_{i=1}^m x_i \cdot V_{k+w_i} = D_{n-m} \cdot B_k$. The correctness then follows from Theorem 5.8 and Corollary 5.9. □

Example 5.11 (Continuation of Example 5.5). In Example 5.5 we have seen that

$$\begin{aligned} I &:= \mathrm{Ann}_{D_3}(f(a,x,b)) \\ &= \langle a\partial_b - \partial_x + 2\partial_a, x\partial_b - \partial_a, x\partial_a + a\partial_a + b\partial_b + 1, x\partial_x - 2x\partial_a - a\partial_a, \\ &\quad b\partial_b^2 + \partial_x\partial_a - \partial_a^2 + \partial_b, a\partial_x\partial_a + 2x\partial_a^2 + a\partial_a^2 + b\partial_x\partial_b + \partial_x + 2\partial_a \rangle \end{aligned}$$

for $f(a,x,b) = \frac{1}{x^2+ax+b}$. Let us compute the restriction module of I with respect to a. So we take $w = (1,0,0)$ and have $b_{I,w}(s) = s$ by using Algorithm 3.16 and hence, $\lambda_0 = 0$, which implies that the restriction module is non-trivial. Using the modification of Algorithm 2.32 mentioned above, we compute the following Gröbner basis with six elements:

$$\begin{aligned} G = \{ & x\partial_b - \partial_a, a\partial_b + 2\partial_a - \partial_x, x\partial_a - x\partial_x - b\partial_b - 1, a\partial_a + x\partial_x + 2b\partial_b + 2, \\ & b\partial_b^2 - \partial_a^2 + \partial_a\partial_x + \partial_b, ax\partial_x + x^2\partial_x + b\partial_x + a + 2x\}. \end{aligned}$$

We read off $(m_1, \ldots, m_6) = (1,1,1,0,2,0)$, where $m_i = \deg_{(-w,w)}(g_i)$. Thus,

$$B_{\lambda_0} = B_{\lambda_0 - m_4} = B_{\lambda_0 - m_6} = B_0 = \{1\}, \qquad B_{\lambda_0 - m_i} = \emptyset \text{ for } i \in \{1,2,3,5\}$$

and

$$\begin{aligned} M' &= \{(a\partial_a + x\partial_x + 2b\partial_b + 2)_{|a=0}, (ax\partial_x + x^2\partial_x + b\partial_x + a + 2x)_{|a=0}\} \\ &= \{x\partial_x + 2b\partial_b + 2, x^2\partial_x + b\partial_x + 2x\}. \end{aligned}$$

Hence, the restriction module with respect to a equals

$$\mathbb{K}\langle x, b, \partial_x, \partial_b \mid \{\partial_x x = x\partial_x + 1, \partial_b b = b\partial_b + 1\}\rangle / \langle x\partial_x + 2b\partial_b + 2, x^2\partial_x + b\partial_x + 2x\rangle.$$

Therefore, the ideal

$$\langle x\partial_x + 2b\partial_b + 2, x^2\partial_x + b\partial_x + 2x\rangle \subseteq \mathrm{Ann}_{D_2}(f(0, x, b))$$

equals the desired answer $(I + \langle x_1, \ldots, x_m\rangle_{D_n}) \cap D_{n-m}$ from Theorem 5.7. However, we can confirm that our result is not the full annihilator. Since in this example, the function f is explicitly given, we can apply Algorithm 5.4 and get

$$\mathrm{Ann}_{D_2}(f(0, x, b)) = \langle 2x\partial_b - \partial_x, x\partial_x + 2b\partial_b + 2, 4b\partial_b^2 + \partial_x^2 + 6\partial_b\rangle.$$

Note that in more complicated cases (i. e. B_{λ_0} contains more than one element), one can get $(I + \langle x_1, \ldots, x_m\rangle_{D_n}) \cap D_{n-m}$ from the restriction module by computing a Gröbner basis with respect to a *position over term ordering* preferring the component belonging to $1 \in B_{\lambda_0}$, see e. g. [Lev05].

5.3 Integration

Integration using D-module theory is closely related to the concept of restriction from the previous section. Let $\mathbb{K}$ be a subfield of $\mathbb{C}$.
For $f_i \in \mathbb{K}[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]$ and $a_i \in \mathbb{C}$, $1 \le i \le p$, consider

$$\int_C \prod_{i=1}^{p} (f_i(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)^{a_i})\, dx_1 \ldots dx_m,$$

where C is an m-dimensional simplex.
Suppose we know a holonomic ideal $I \subseteq D_n$ such that

$$I \subseteq \mathrm{Ann}_{D_n}(\prod_{i=1}^{p} f_i^{a_i}).$$

Let D_{n-m} denote the $(n-m)$-th Weyl algebra

$$\mathbb{K}\langle x_{m+1}, \ldots, x_n, \partial_{m+1}, \ldots, \partial_n \mid \{\partial_i x_j = x_j\partial_i + \delta_{ij} \mid m+1 \le i, j \le n\}\rangle.$$

Our goal now is to compute an ideal $J \subseteq D_{n-m}$ such that

$$J \subseteq \mathrm{Ann}_{D_{n-m}} \left(\int\limits_C \prod_{i=1}^{p} (f_i(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)^{a_i})\, dx_1 \ldots dx_m \right)$$

directly from I.

Remark 5.12. Let $I \subseteq D_n$ be a left ideal. Analogously to Remark 5.6 consider

$$D_n/\langle \partial_1, \ldots, \partial_m \rangle_{D_n} \otimes_{D_n} D_n/I \cong D_n/(I + \langle \partial_1, \ldots, \partial_m \rangle_{D_n}) =: I_m.$$

The left D_{n-m}-module I_m is called the *integral module* of D_n/I or simply the *integral* of D_n/I with respect to $x_1, \ldots, x_m$.

Theorem 5.13 ([SST00, Theorem 5.5.1]). Let $a_i \in \mathbb{C}$, $1 \leq i \leq p$ and $I \subseteq \mathrm{Ann}_{D_n}(\prod_{i=1}^p f_i^{a_i})$ be a holonomic ideal in D_n. Then

$$(I + \langle \partial_1, \ldots, \partial_m \rangle_{D_n}) \cap D_{n-m}$$

is a left ideal in D_{n-m} annihilating $\int_C (\prod_{i=1}^p f_i^{a_i})\, dx_1 \ldots dx_m$.

Evidently, all we need now is a link between the integral and the restriction module. Consider the *Fourier transform* with respect to $x_1, \ldots, x_m$,

$$\mathcal{F}_m : D_n \to D_n, \begin{cases} x_i \mapsto -\partial_i, & \partial_i \mapsto x_i \quad 1 \leq i \leq m, \\ x_i \mapsto x_i, & \partial_i \mapsto \partial_i \quad m+1 \leq i \leq n. \end{cases}$$

The Fourier transform is an automorphism of the Weyl algebra fulfilling

$$(I + \langle \partial_1, \ldots, \partial_m \rangle_{D_n}) = \mathcal{F}_m^{-1}(\mathcal{F}_m(I) + \langle x_1, \ldots, x_m \rangle_{D_n}).$$

Therefore, using the notations from Algorithm 5.10, the integral module can be computed as follows.

Algorithm 5.14 (`integralModule`).
Input: $I \subseteq D_n$ holonomic
Output: a presentation D_{n-m}^k/M of the integral module
  $R_m \cong (D_{n-m} \cdot B_{\lambda_0})/(D_{n-m} \cdot M')$ the restriction module of $\mathcal{F}_m(I)$ $\to$ Algorithm 5.10
  return $(\mathcal{F}_m^{-1}(D_{n-m} \cdot B_{\lambda_0}))/(\mathcal{F}_m^{-1}(D_{n-m} \cdot M'))$

Note that analogously to the restriction algorithm, one can get $(I + \langle \partial_1, \ldots, \partial_m \rangle_{D_n}) \cap D_{n-m}$ from the integral module by computing a Gröbner basis with respect to a *position over term ordering* preferring the component belonging to $1 \in \mathcal{F}_m^{-1}(B_{\lambda_0})$.

Example 5.15 (Continuation of Example 5.11). In Example 5.11 we have computed

$$I := \langle x\partial_x + 2b\partial_b + 2, x^2\partial_x + b\partial_x + 2x\rangle \subseteq \mathrm{Ann}_{D_2}(f(0,x,b)),$$

where $D_2 = \mathbb{K}\langle x, b, \partial_x, \partial_b \mid \{\partial_x x = x\partial_x + 1, \partial_b b = b\partial_b + 1\}\rangle$ and $f(0,x,b) = \frac{1}{x^2+b}$. Let us now compute the integral module of I with respect to x. Applying the Fourier transform to the generators of I yields

$$\mathcal{F}_1(x\partial_x + 2b\partial_b + 2) = -\partial_x x + 2b\partial_b + 2 = -x\partial_x + 2b\partial_b + 1 \quad \text{and}$$
$$\mathcal{F}_1(x^2\partial_x + b\partial_x + 2x) = \partial_x^2 x + bx - 2\partial_x = x\partial_x^2 + xb.$$

We have $b_{\mathcal{F}_1(I),w}(s) = s(s-1)$ for $w = (1,0)$ by using Algorithm 3.16 and hence, $\lambda_0 = 1$, which implies that the restriction module of $\mathcal{F}_1(I)$ is non-trivial. Proceeding as in Example 5.11 we get

$$G = \{x\partial_x - 2b\partial_b - 1, 2b\partial_x\partial_b + xb, 4b^2\partial_b^2 + x^2b + 6b\partial_b\},$$
$$m_1 = 0, \quad m_2 = 1, \quad m_3 = 0,$$
$$B_{\lambda_0} = B_{\lambda_0 - m_1} = B_{\lambda_0 - m_3} = B_1 = \{1, \partial_x\}, \quad B_{\lambda_0 - m_2} = B_0 = \{1\}$$

and

$$\begin{aligned} M' &= G_{|x=0} \cup \{(\partial_x \cdot (x\partial_x - 2b\partial_b - 1))_{|x=0}, (\partial_x \cdot (4b^2\partial_b^2 + x^2b + 6b\partial_b))_{|x=0}\} \\ &= \{-2b\partial_b - 1, 2b\partial_x\partial_b, 4b^2\partial_b^2 + 6b\partial_b, -2b\partial_x\partial_b, 4b^2\partial_x\partial_b^2 + 6b\partial_x\partial_b\}. \end{aligned}$$

Applying the inverse of $\mathcal{F}_1$, we have

$$\mathcal{F}_1^{-1}(M') = \{-2b\partial_b - 1, -2bx\partial_b, 4b^2\partial_b^2 + 6b\partial_b, 2bx\partial_b, -4b^2x\partial_b^2 - 6bx\partial_b\}$$

and therefore we obtain the integral module of I with respect to x as follows:

$$I_1 := (D_1 \cdot \{1, x\})/(D_1 \cdot \{2b\partial_b + 1, bx\partial_b, 2b^2\partial_b^2 + 3b\partial_b\}),$$

where $D_1 = \mathbb{K}\langle b, \partial_b \mid \partial_b b = b\partial_b + 1\rangle$. In terms of matrix representations, we have

$$I_1 \cong D_1^2 \Big/ D_1^2 \cdot \begin{bmatrix} b\partial_b + \frac{1}{2} & 0 \\ 0 & b\partial_b \end{bmatrix} \cong D_1/\langle b\partial_b + \tfrac{1}{2}\rangle \oplus D_1/\langle b\partial_b\rangle.$$

Therefore, the ideal

$$\langle (b\partial_b + \tfrac{1}{2}) \cdot b\partial_b\rangle = \langle b^2\partial_b^2 + \tfrac{3}{2}b\partial_b\rangle$$

equals the desired answer $(I + \langle \partial_1, \ldots, \partial_m\rangle_{D_n}) \cap D_{n-m}$ from Theorem 5.13.

5.4 Integration using the Bernstein operator

Let $f \in \mathbb{K}[x_1, \dots, x_n]$ and C be an n-dimensional simplex in $\mathbb{K}^n$. By Equation (4.1)

$$\zeta(s) := \int_C f(x)^s dx = \frac{1}{b_f(s)} \int_C P \bullet f(x)^{s+1} dx$$

for a b-operator $P \in D_n[s]$. The so-called *Igusa zeta-function* $\zeta(s)$ is given in terms of recurrence equations with polynomial coefficients (in other words, as the annihilator ideal in the shift algebra in the variables s and E_s, where the latter denotes the shift operator with respect to s). In some cases it is possible to compute a closed form solution to $\zeta(s)$, starting from the annihilator ideal and some initial data (e. g. by using MAPLE or MATHEMATICA).

Example 5.16. Let $f = x^2 - x \in \mathbb{R}[x]$. Then the Bernstein operator reads as $P = (2x-1)\partial_x - 4(s+1)$ and $b_f(s) = s+1$. Any simplex in $\mathbb{R}^1$ is an interval $[a, b] =: C$.

$$\begin{aligned}
\zeta(s) &= \int_C f(x)^s dx = \frac{1}{b_f(s)} \int_C P \bullet f(x)^{s+1} dx \\
&= \frac{1}{s+1} \int_C ((2x-1)\partial_x - 4(s+1)) \bullet f(x)^{s+1} dx \\
&= \frac{1}{s+1} \int_C \left((2x-1)\partial_x \bullet f(x)^{s+1}\right) - \left(4(s+1) \bullet f(x)^{s+1}\right) dx \\
&= \frac{1}{s+1} \left(\int_C (2x-1)\partial_x \bullet f(x)^{s+1} dx \right) - 4\zeta(s+1).
\end{aligned}$$

By partial integration,

$$\int_C (2x-1)(\partial_x \bullet f(x)^{s+1}) dx = (2x-1)f(x)^{s+1} \mid_C -2 \int_C f(x)^{s+1} dx,$$

hence

$$\zeta(s) = \frac{1}{s+1} \cdot (2x-1)f(x)^{s+1} \mid_C - \frac{2}{s+1}\zeta(s+1) - 4\zeta(s+1),$$

and thus

$$(4s+6)\zeta(s+1) + (s+1)\zeta(s) = (2b-1)(b^2-b)^{s+1} - (2a-1)(a^2-a)^{s+1}.$$

The right hand side, say $R(s)$, satisfies the homogeneous recurrence

$$R(s+2) - (a^2 - a + b^2 - b)R(s+1) + (a^2-a)(b^2-b)R(s) = 0$$

of order 2. Substituting the left hand side into it, we obtain a homogeneous recurrence with polynomial coefficients of order 3:

$$\begin{aligned}
&\big((a^2-a)(b^2-b)(s+1)\big)\cdot\zeta(s)\\
-\big((s+2)(a^2-a+b^2-b)-(4s+6)(a^2-a)(b^2-b)\big)&\cdot\zeta(s+1)\\
-\big((4s+10)(a^2-a+b^2-b)-(s+3)\big)&\cdot\zeta(s+2)\\
+(4s+14)&\cdot\zeta(s+3)=0.
\end{aligned}$$

To ensure the uniqueness of a solution to this equation, we need to specify three initial values, which can be easily done. However, it is not guaranteed that such recurrences admit a closed form solution. Thus, the information about $\zeta(s)$ is contained in the recurrence itself.

5.5 Further applications

Further applications of the theory of b-functions include related invariants and other topics. It is beyond the scope of this work to explain them in detail. Nevertheless, we would like to briefly mention them as these applications are subjects of ongoing research. We refer to the given literature.

- The eigenvalues of a *local monodromy* [Mil68] correspond to the roots of a certain b-function [Mal75, Mal83].
- Certain *spectral numbers* [Ste77, Var81] are roots of the Bernstein-Sato polynomial [GH07, HS99, Sai93].
- There is a conjecture stating that every pole of the *topological zeta-function* is a root of the Bernstein-Sato polynomial [Loe88, Vey06].
- The concept of Bernstein-Sato polynomials for hypersurfaces examined in this work can be generalized to *arbitrary varieties* [BMS06]. In the *affine algebraic* case it is possible to compute the b-function of a variety with the methods and algorithms presented in this work [ALMM09].
- Recently, an algorithm to compute *jumping coefficients* and their corresponding *multiplier ideals* was published, using the concept of Bernstein-Sato polynomials for affine varieties [Shi08].
- There are algorithms, based on D-module theory, for computing certain *de Rham cohomologies* [OT99, SST00, Wal00], which involves the computation of the *Čech complex* in the realm of D-modules. These algorithms require computations of s-parametric annihilators and finding the corresponding minimal integer roots of Bernstein-Sato polynomials [Wal99, Wal02, OT01].

6 Experiments and implementation

In this chapter, we give timings achieved with our implementation of the algorithms we have discussed in this work.

6.1 Implementation

We briefly describe the main procedures along with some of their features, which we have implemented in SINGULAR in one of the libraries `bfun.lib`, `dmodapp.lib` or `dmod.lib`, which are freely distributed together with SINGULAR. Note that by loading one of these libraries, the other ones are loaded automatically. We refer to the SINGULAR manual[1] for an explicit description on how to call the procedures.

`initialIdealW`

For an ideal $I \subseteq D_n$ and a pair of weights $0 \neq (u, v) \in \mathbb{R}^{2n}$ satisfying $0 \leq_{\mathrm{cw}} u + v$, `initialIdealW` computes $\mathrm{in}_{(u,v)}(I)$ according to Algorithm 2.32. A vector used for homogenization can be specified via an optional argument, i. e. if not given, $(1, \ldots, 1)$ is used as homogenization weight.

Note that the additional "`W`" in name of the SINGULAR procedure stands for "Weyl". The reason for using it is an already existing procedure `initialIdeal` from `tropical.lib`, which uses techniques that do not work in the non-commutative setting.

`initialMalgrange`

For $f \in \mathbb{K}[x_1, \ldots, x_n]$, `initialMalgrange` computes the initial ideal of the Malgrange ideal of f with respect to the weight $(-w, w)$, where $w = (1, 0, \ldots, 0)$, according to Algorithm 2.32. Homogenization weights as proposed in Remark 4.2 are being used.

`SannfsBFCT`

For $f \in \mathbb{K}[x_1, \ldots, x_n]$, `SannfsBFCT` computes a Gröbner basis of $\mathrm{Ann}_{D_n[s]}(f^s) + \langle f \rangle$ following Algorithm 4.22, i. e. the syzygy-driven enhancement of the approach by Briançon-Maisonobe. Also, an *anti-elimination ordering* for s is used, i. e. in contrast to an elimination ordering for s, we choose s to be greater than any other variable, see Section 6.2.6.

[1] `http://www.singular.uni-kl.de/Manual/latest/`

Note that by specifying an optional argument, `SannfsBFCT` computes a Gröbner basis of $\mathrm{Ann}_{D_n[s]}(f^s) + \langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle$, see Lemma 4.18.

`linReduce`

Given a G-algebra A, an ideal $I \subseteq A$ and a polynomial $f \in A$, `linReduce` reduces f with the generators of I by solely using linear reductions (no monomial multiplications) according to Algorithm 3.13.

`pIntersect`

For a G-algebra A, an ideal $J \subseteq A$ given as Gröbner basis and a polynomial $s \in A$, `pIntersect` computes the monic generator of $J \cap \mathbb{K}[s]$ according to Algorithm 3.12 with the enhancement given in Section 3.3.1.

The necessary condition for the nontriviality of the intersection from Lemma 3.7 is checked first. Moreover, a degree bound can be specified as an optional argument.

`bfct`

For $f \in \mathbb{K}[x_1, \ldots, x_n]$, `bfct` computes the Bernstein-Sato polynomial of f by calling `initialMalgrange` and `pIntersect`. An ordering as proposed in Remark 4.4 (with `valvars` from `presolve.lib` [Gre10] as valuation function) is used.

`bfctAnn`

For $f \in \mathbb{K}[x_1, \ldots, x_n]$, `bfctAnn` computes the Bernstein-Sato polynomial of f by calling `SannfsBFCT` (with the optional parameter to add all partial derivatives of f as mentioned above) and `pIntersect`.

`bfctIdeal`

For an ideal $I \subseteq D_n$ and a weight $0 \neq w \in \mathbb{R}^n_{\geq 0}$ `bfctIdeal` computes the global b-function of I with respect to w according to Algorithm 3.16 by calling `initialIdealW` and `pIntersect`.

Note that if I is not holonomic, the termination of Algorithm 3.12 cannot be guaranteed. In this case, a warning message is printed and `pIntersect` is interrupted if no solution of degree less than or equal to ten is found.

6.2 Experiments

6.2.1 Examples

We consider the examples given in Tables 6.1 and 6.2. We distinguish between hyperplane arrangements and other kinds of polynomials. The hyperplane arrangements used were proposed to us by Uli Walther. The non-hyperplane arrangements are selected examples. Although “easy looking”, they are intrinsic for different reasons. They represent families of polynomials with “bad” singularities.

Tables 6.3 and 6.4 show the Bernstein-Sato polynomials of the corresponding examples computed by our implementation. They are given by the negatives of their roots. Note that multiple roots appear more than once according to their multiplicity.

Example	Input
uw1	$-xyz(y-z)(y+z)$
uw2	$-xyz(x+y+z)(y-z)$
uw3	$-xyz(x+z)(y-z)$
uw4	$xyz(x+y+z)(3x+2y+z)$
uw5	$-xyz(y-z)(2y+z)(y+z)$
uw6	$-xyz(x+2y+z)(y-z)(y+z)$
uw7	$(x-z)xyz(y-z)(y+z)$
uw8	$-xyz(x-2y-z)(x-y+2z)(y+z)$
uw9	$-xyz(x-y+2z)(2x-y-z)(y+z)$
uw10	$xyz(-x-y+z)(x-y+z)(y+z)$
uw11	$(x-z)xyz(-x+y)(y+z)$
uw12	$-(x-z)xyz(-x+y)(y-z)$
uw13	$xyz(4x+2y+z)(9x+3y+z)(x+y+z)$
uw14	$xyz(2y+z)(y+z)(4y+z)(3y+z)$
uw15	$-xyz(-x+y-z)(3y+z)(2y+z)(y+z)$
uw16	$-(x-z)xyz(2y+z)(3y+z)(y+z)$
uw17	$xyz(-x-y+z)(x-2y+z)(y+z)(2y+z)$
uw18	$-(x-z)xyz(2x+2y-z)(y-z)(y+z)$
uw19	$-xyz(-x+y+2z)(x+y+2z)(y-z)(y+z)$
uw20	$(x-z)xyz(x+z)(y-z)(y+z)$
uw21	$-(x-z)xyz(-x+y-z)(y-z)(y+z)$
uw22	$-xyz(x+z)(-x+y)(y-z)(y+z)$
uw23	$xyz(-x-y+z)(3x-2y+z)(4x+2y+z)(y+z)$
uw24	$(x-z)xyz(-3x-y+z)(4x-2y-z)(y+z)$
uw25	$(x-z)xyz(-x+y)(3x+2y+z)(y+z)$
uw26	$(x-z)xyz(x-y+z)(2x-2y-z)(y+z)$
uw27	$xyz(x+y)(-x+2y+z)(x+y+z)(y+z)$
uw28	$xyz(x+z)(-x+y+z)(x+y)(y+z)$
uw29	$-xyz(x+z)(x+y)(3x+y-2z)(y+z)$
uw30	$-xyz(x-2z)(x-y-z)(x-y)(y+z)$
uw31	$-xyz(2x-z)(x-y-z)(x-y)(y+z)$
uw32	$(x-z)xyz(x-y-z)(x-y)(y-z)$
uw33	$xyz(x+y+z)(9x+3y+z)(16x+4y+z)(4x+2y+z)$

Table 6.1: Hyperplane arrangements

Example	Input
ab23	$(z^2+w^3)(2zx+3w^2y)$
chal2	$(x^3+y^2)(y^3+x^2)$
chal3	$(x^2+y^2(1+y))(y^3+x^2)$
chal3b	$(x^2+y^2(1+y))(x^3+y^2)$
chal4	$(y^5+xy^4+x^4)(x^5+x^4y+y^4)$
cnu3	$x^4z-xy^3z+x^3y-y^4$
cnu4	$x^5z-xy^4z+x^4y-y^5$
cnu5	$x^6z-xy^5z+x^5y-y^6$
cnu6	$x^7z-xy^6z+x^6y-y^7$
cnu7s1	$(xz+y)(x^7-y^7)$
cusp23cusp32	$(x^2+y^3)(x^3+y^2)$
cusp34	x^3+y^4
reiffen45	$x^4+y^5+xy^4$
reiffen46	$x^4+y^6+xy^5$
reiffen47	$x^4+y^7+xy^6$
reiffen48	$x^4+y^8+xy^7$
reiffen49	$x^4+y^9+xy^8$
reiffen56	$x^5+y^6+xy^5$
reiffen57	$x^5+y^7+xy^6$
reiffen58	$x^5+y^8+xy^7$
reiffen59	$x^5+y^9+xy^8$
reiffen66	$x^6+y^6+xy^5$
reiffen67	$x^6+y^7+xy^6$
reiffen68	$x^6+y^8+xy^7$
reiffen69	$x^6+y^9+xy^8$
reiffen77	$x^7+y^7+xy^6$
reiffen78	$x^7+y^8+xy^7$
reiffen79	$x^7+y^9+xy^8$
reiffen88	$x^8+y^8+xy^7$
reiffen89	$x^8+y^9+xy^8$
reiffen99	$x^9+y^9+xy^8$
reiffen11	$x^{11}+y^{11}+xy^{10}$
tt32	$x^3+y^3+z^3-(xyz)^2$
tt42	$x^4+y^4+z^4-(xyz)^2$
tt43	$x^4+y^4+z^4-(xyz)^3$
xyzcusp45	$(xy+z)(y^4+z^5+yz^4)$
xyzReiffen45	$xy^5z+y^6+x^5z+x^4y$

Table 6.2: Various polynomials

Example	Negatives of the roots of the Bernstein-Sato polynomial
uw1	$\frac{3}{2}, \frac{5}{4}, 1, 1, 1, \frac{3}{4}, \frac{1}{2}$
uw2	$\frac{8}{5}, \frac{7}{5}, \frac{4}{3}, \frac{6}{5}, 1, 1, 1, \frac{4}{5}, \frac{2}{3}, \frac{3}{5}$
uw3	$\frac{7}{5}, \frac{4}{3}, \frac{6}{5}, 1, 1, 1, \frac{4}{5}, \frac{2}{3}, \frac{3}{5}$
uw4	$\frac{8}{5}, \frac{7}{5}, \frac{6}{5}, 1, 1, 1, \frac{4}{5}, \frac{3}{5}$
uw5	$\frac{8}{5}, \frac{7}{5}, \frac{6}{5}, 1, 1, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}$
uw6	$\frac{5}{3}, \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}$
uw7	$\frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{5}{4}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{3}{4}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2}$
uw8	$\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}$
uw9	$\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}$
uw10	$\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}$
uw11	$\frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}$
uw12	$\frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}$
uw13	$\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}$
uw14	$\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, \frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$
uw15	$\frac{12}{7}, \frac{8}{5}, \frac{11}{7}, \frac{10}{7}, \frac{7}{5}, \frac{9}{7}, \frac{6}{5}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{4}{5}, \frac{5}{7}, \frac{3}{5}, \frac{4}{7}, \frac{3}{7}, \frac{2}{5}$
uw16	$\frac{8}{5}, \frac{11}{7}, \frac{10}{7}, \frac{7}{5}, \frac{4}{3}, \frac{9}{7}, \frac{6}{5}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{4}{5}, \frac{5}{7}, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{3}{7}, \frac{2}{5}$
uw17	$\frac{12}{7}, \frac{11}{7}, \frac{3}{2}, \frac{10}{7}, \frac{9}{7}, \frac{5}{4}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{3}{4}, \frac{5}{7}, \frac{4}{7}, \frac{1}{2}, \frac{3}{7}$
uw18	$\frac{12}{7}, \frac{11}{7}, \frac{3}{2}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{5}{4}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{3}{4}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}, \frac{3}{7}$
uw19	$\frac{12}{7}, \frac{11}{7}, \frac{3}{2}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{5}{4}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{3}{4}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}, \frac{3}{7}$
uw20	$\frac{11}{7}, \frac{3}{2}, \frac{10}{7}, \frac{9}{7}, \frac{5}{4}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{3}{4}, \frac{5}{7}, \frac{4}{7}, \frac{1}{2}, \frac{3}{7}$
uw21	$\frac{11}{7}, \frac{3}{2}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{5}{4}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{3}{4}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}, \frac{3}{7}$
uw22	$\frac{11}{7}, \frac{3}{2}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{5}{4}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{3}{4}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{1}{2}, \frac{3}{7}$
uw23	$\frac{12}{7}, \frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw24	$\frac{12}{7}, \frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw25	$\frac{12}{7}, \frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw26	$\frac{12}{7}, \frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw27	$\frac{12}{7}, \frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw28	$\frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw29	$\frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw30	$\frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw31	$\frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw32	$\frac{11}{7}, \frac{10}{7}, \frac{4}{3}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{2}{3}, \frac{4}{7}, \frac{3}{7}$
uw33	$\frac{12}{7}, \frac{11}{7}, \frac{10}{7}, \frac{9}{7}, \frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{4}{7}, \frac{3}{7}$

Table 6.3: Hyperplane arrangements: Bernstein-Sato polynomials given by the negatives of their roots

Example	Negatives of the roots of the Bernstein-Sato polynomial
ab23	$\frac{13}{9}, \frac{7}{5}, \frac{11}{9}, \frac{6}{5}, \frac{10}{9}, 1, 1, 1, \frac{8}{9}, \frac{4}{5}, \frac{7}{9}, \frac{3}{5}, \frac{5}{9}$
chal2	$\frac{13}{10}, \frac{11}{10}, 1, 1, \frac{9}{10}, \frac{7}{10}, \frac{1}{2}, \frac{1}{2}$
chal3	$\frac{13}{10}, \frac{5}{4}, \frac{11}{10}, 1, 1, \frac{9}{10}, \frac{3}{4}, \frac{7}{10}, \frac{1}{2}, \frac{1}{2}$
chal3b	$\frac{13}{10}, \frac{5}{4}, \frac{11}{10}, 1, 1, \frac{9}{10}, \frac{3}{4}, \frac{7}{10}, \frac{1}{2}, \frac{1}{2}$
cnu3	$\frac{5}{4}, 1, 1, 1, \frac{3}{4}, \frac{1}{2}$
cnu4	$\frac{6}{5}, 1, 1, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}$
cnu5	$\frac{7}{6}, 1, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$
cnu6	$\frac{8}{7}, 1, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{4}{7}, \frac{3}{7}, \frac{2}{7}$
cnu7s1	$\frac{9}{8}, 1, 1, 1, \frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}$
cusp23cusp32	$\frac{13}{10}, \frac{11}{10}, 1, 1, \frac{9}{10}, \frac{7}{10}, \frac{1}{2}, \frac{1}{2}$
cusp34	$\frac{17}{12}, \frac{7}{6}, \frac{13}{12}, 1, \frac{11}{12}, \frac{5}{6}, \frac{7}{12}$
reiffen45	$\frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{7}{10}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, 1, \frac{21}{20}, \frac{11}{10}, \frac{23}{20}, \frac{13}{10}, \frac{27}{20}$
reiffen46	$\frac{5}{12}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}, 1, 1, \frac{13}{12}, \frac{7}{6}, \frac{5}{4}, \frac{4}{3}, \frac{17}{12}$
reiffen47	$\frac{11}{28}, \frac{15}{28}, \frac{17}{28}, \frac{9}{14}, \frac{19}{28}, \frac{11}{14}, \frac{23}{28}, \frac{25}{28}, \frac{13}{14}, \frac{27}{28}, 1, \frac{29}{28}, \frac{15}{14}, \frac{31}{28}, \frac{33}{28}, \frac{17}{14}, \frac{37}{28}, \frac{19}{14}, \frac{41}{28}$
reiffen48	$\frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1, 1, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}$
reiffen49	$\frac{13}{36}, \frac{17}{36}, \frac{7}{12}, \frac{11}{18}, \frac{23}{36}, \frac{25}{36}, \frac{13}{18}, \frac{29}{36}, \frac{5}{6}, \frac{31}{36}, \frac{11}{12}, \frac{17}{18}, \frac{35}{36}, 1, \frac{37}{36}, \frac{19}{18}, \frac{13}{12}, \frac{41}{36}, \frac{7}{6}, \frac{43}{36}, \frac{23}{18}, \frac{47}{36}, \frac{25}{18}, \frac{17}{12}, \frac{55}{36}$
reiffen56	$\frac{11}{30}, \frac{13}{30}, \frac{7}{15}, \frac{8}{15}, \frac{17}{30}, \frac{19}{30}, \frac{7}{10}, \frac{11}{15}, \frac{23}{30}, \frac{13}{15}, \frac{9}{10}, \frac{14}{15}, \frac{29}{30}, 1, \frac{31}{30}, \frac{16}{15}, \frac{11}{10}, \frac{17}{15}, \frac{37}{30}, \frac{19}{15}, \frac{13}{10}$
reiffen57	$\frac{12}{35}, \frac{16}{35}, \frac{17}{35}, \frac{18}{35}, \frac{19}{35}, \frac{22}{35}, \frac{23}{35}, \frac{24}{35}, \frac{26}{35}, \frac{27}{35}, \frac{29}{35}, \frac{31}{35}, \frac{32}{35}, \frac{33}{35}, \frac{34}{35}, 1, \frac{36}{35}, \frac{37}{35}, \frac{38}{35}, \frac{39}{35}, \frac{41}{35}, \frac{43}{35}, \frac{44}{35}, \frac{46}{35}, \frac{48}{35}$
reiffen58	$\frac{13}{40}, \frac{9}{20}, \frac{19}{40}, \frac{21}{40}, \frac{11}{20}, \frac{23}{40}, \frac{13}{20}, \frac{27}{40}, \frac{7}{10}, \frac{29}{40}, \frac{31}{40}, \frac{33}{40}, \frac{17}{20}, \frac{9}{10}, \frac{37}{40}, \frac{19}{20}, \frac{39}{40}, 1, \frac{41}{40}, \frac{21}{20}, \frac{43}{40}, \frac{11}{10}, \frac{23}{20}, \frac{47}{40}, \frac{49}{40}, \frac{51}{40}, \frac{13}{10}, \frac{27}{20}, \frac{57}{40}$
reiffen59	$\frac{14}{45}, \frac{19}{45}, \frac{22}{45}, \frac{23}{45}, \frac{8}{15}, \frac{26}{45}, \frac{28}{45}, \frac{29}{45}, \frac{31}{45}, \frac{32}{45}, \frac{11}{15}, \frac{34}{45}, \frac{37}{45}, \frac{38}{45}, \frac{13}{15}, \frac{41}{45}, \frac{14}{15}, \frac{43}{45}, \frac{44}{45}, 1, \frac{46}{45}, \frac{47}{45}, \frac{16}{15}, \frac{49}{45}, \frac{17}{15}, \frac{52}{45}, \frac{53}{45}, \frac{56}{45}, \frac{19}{15}, \frac{58}{45}, \frac{61}{45}, \frac{62}{45}, \frac{22}{15}$
reiffen66	$\frac{5}{3}, \frac{3}{2}, \frac{4}{3}, \frac{7}{6}, 1, 1, \frac{5}{6}, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}$
reiffen67	$\frac{13}{42}, \frac{5}{14}, \frac{8}{21}, \frac{17}{42}, \frac{19}{42}, \frac{10}{21}, \frac{11}{21}, \frac{23}{42}, \frac{25}{42}, \frac{13}{21}, \frac{9}{14}, \frac{29}{42}, \frac{31}{42}, \frac{16}{21}, \frac{11}{14}, \frac{17}{21}, \frac{37}{42}, \frac{19}{21}, \frac{13}{14}, \frac{20}{21}, \frac{41}{42}, 1, \frac{43}{42}, \frac{22}{21}, \frac{15}{14}, \frac{23}{21}, \frac{47}{42}, \frac{25}{21}, \frac{17}{14}, \frac{26}{21}, \frac{53}{42}$
reiffen68	$\frac{7}{24}, \frac{3}{8}, \frac{5}{12}, \frac{11}{24}, \frac{13}{24}, \frac{7}{12}, \frac{5}{8}, \frac{2}{3}, \frac{17}{24}, \frac{3}{4}, \frac{19}{24}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{23}{24}, 1, 1, \frac{25}{24}, \frac{13}{12}, \frac{9}{8}, \frac{7}{6}, \frac{29}{24}, \frac{5}{4}, \frac{31}{24}, \frac{4}{3}$
reiffen69	$\frac{5}{18}, \frac{7}{18}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{11}{18}, \frac{2}{3}, \frac{13}{18}, \frac{7}{9}, \frac{5}{6}, \frac{8}{9}, \frac{17}{18}, 1, 1, \frac{19}{18}, \frac{10}{9}, \frac{7}{6}, \frac{11}{9}, \frac{23}{18}, \frac{4}{3}, \frac{25}{18}$
reiffen77	$\frac{12}{7}, \frac{11}{7}, \frac{10}{7}, \frac{9}{7}, \frac{8}{7}, 1, 1, \frac{6}{7}, \frac{5}{7}, \frac{4}{7}, \frac{3}{7}, \frac{2}{7}$
reiffen78	$\frac{15}{56}, \frac{17}{56}, \frac{9}{28}, \frac{19}{56}, \frac{5}{14}, \frac{11}{28}, \frac{23}{56}, \frac{25}{56}, \frac{13}{28}, \frac{27}{56}, \frac{29}{56}, \frac{15}{28}, \frac{31}{56}, \frac{33}{56}, \frac{17}{28}, \frac{9}{14}, \frac{37}{56}, \frac{19}{28}, \frac{39}{56}, \frac{41}{56}, \frac{43}{56}, \frac{11}{14}, \frac{45}{56}, \frac{23}{28}, \frac{47}{56}, \frac{25}{28}, \frac{51}{56}, \frac{13}{14}, \frac{53}{56}, \frac{27}{28}, \frac{55}{56}, 1, \frac{57}{56}, \frac{29}{28}, \frac{59}{56}, \frac{15}{14}, \frac{61}{56}, \frac{31}{28}, \frac{65}{56}, \frac{33}{28}, \frac{67}{56}, \frac{17}{14}, \frac{69}{56}$
reiffen79	$\frac{16}{63}, \frac{20}{63}, \frac{22}{63}, \frac{23}{63}, \frac{8}{21}, \frac{25}{63}, \frac{26}{63}, \frac{29}{63}, \frac{10}{21}, \frac{31}{63}, \frac{32}{63}, \frac{11}{21}, \frac{34}{63}, \frac{37}{63}, \frac{38}{63}, \frac{13}{21}, \frac{40}{63}, \frac{41}{63}, \frac{43}{63}, \frac{44}{63}, \frac{46}{63}, \frac{47}{63}, \frac{16}{21}, \frac{50}{63}, \frac{17}{21}, \frac{52}{63}, \frac{53}{63}, \frac{55}{63}, \frac{19}{21}, \frac{58}{63}, \frac{59}{63}, \frac{20}{21}, \frac{61}{63}, \frac{62}{63}, 1, \frac{64}{63}, \frac{65}{63}, \frac{22}{21}, \frac{67}{63}, \frac{68}{63}, \frac{23}{21}, \frac{71}{63}, \frac{73}{63}, \frac{74}{63}, \frac{25}{21}, \frac{76}{63}, \frac{26}{21}, \frac{80}{63}, \frac{82}{63}$
reiffen88	$\frac{7}{4}, \frac{13}{8}, \frac{3}{2}, \frac{11}{8}, \frac{5}{4}, \frac{9}{8}, 1, 1, \frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}$
reiffen89	$\frac{17}{72}, \frac{19}{72}, \frac{5}{18}, \frac{7}{24}, \frac{11}{36}, \frac{23}{72}, \frac{25}{72}, \frac{13}{36}, \frac{7}{18}, \frac{29}{72}, \frac{5}{12}, \frac{31}{72}, \frac{11}{24}, \frac{17}{36}, \frac{35}{72}, \frac{37}{72}, \frac{19}{36}, \frac{13}{24}, \frac{41}{72}, \frac{7}{12}, \frac{43}{72}, \frac{11}{18}, \frac{23}{36}, \frac{47}{72}, \frac{49}{72}, \frac{25}{36}, \frac{17}{24}, \frac{13}{18}, \frac{53}{72}, \frac{55}{72}, \frac{19}{24}, \frac{29}{36}, \frac{59}{72}, \frac{5}{6}, \frac{61}{72}, \frac{31}{36}, \frac{65}{72}, \frac{11}{12}, \frac{67}{72}, \frac{17}{18}, \frac{23}{24}, \frac{35}{36}, \frac{71}{72}, 1, \frac{73}{72}, \frac{37}{36}, \frac{25}{24}, \frac{19}{18}, \frac{77}{72}, \frac{13}{12}, \frac{79}{72}, \frac{41}{36}, \frac{83}{72}, \frac{7}{6}, \frac{85}{72}, \frac{43}{36}, \frac{29}{24}$
reiffen99	$\frac{16}{9}, \frac{5}{3}, \frac{14}{9}, \frac{13}{9}, \frac{4}{3}, \frac{11}{9}, \frac{10}{9}, 1, 1, \frac{8}{9}, \frac{7}{9}, \frac{2}{3}, \frac{5}{9}, \frac{4}{9}, \frac{1}{3}, \frac{2}{9}$
reiffen11	$\frac{20}{11}, \frac{19}{11}, \frac{18}{11}, \frac{17}{11}, \frac{16}{11}, \frac{15}{11}, \frac{14}{11}, \frac{13}{11}, \frac{12}{11}, 1, 1, \frac{10}{11}, \frac{9}{11}, \frac{8}{11}, \frac{7}{11}, \frac{6}{11}, \frac{5}{11}, \frac{4}{11}, \frac{3}{11}, \frac{2}{11}$
tt32	$2, \frac{5}{3}, \frac{4}{3}, 1, 1$
tt42	$2, \frac{7}{4}, \frac{3}{2}, \frac{5}{4}, 1, 1, \frac{3}{4}$
tt43	$\frac{9}{4}, 2, \frac{7}{4}, \frac{3}{2}, \frac{5}{4}, 1, 1, \frac{3}{4}$
xyzcusp45	$\frac{31}{24}, \frac{5}{4}, \frac{29}{24}, \frac{9}{8}, \frac{13}{12}, \frac{25}{24}, 1, 1, \frac{23}{24}, \frac{11}{12}, \frac{7}{8}, \frac{19}{24}, \frac{3}{4}, \frac{17}{24}, \frac{5}{8}, \frac{7}{12}, \frac{13}{24}, \frac{11}{24}, \frac{5}{12}, \frac{3}{8}$

Table 6.4: Various polynomials: Bernstein-Sato polynomials given by the negatives of their roots

6.2.2 Comparisons to other systems

We compare our implementation with the existing ones in Asir [NST06] by Masayuki Noro [Nor02] and Macaulay 2 [GS05] by Harrison Tsai and Anton Leykin [TL06].

We request the computation of Bernstein-Sato polynomials with the procedures available in the particular system and measure the time.

All timings are given in the format "[hours]:minutes:seconds".

We use the shorthand notations $t^{\times}$ when we have cancelled the process after the time t and $t^{\dagger}$ when the process ran out of memory after the time t.

An entry "n/a" means that we have not the requested the particular computation.

The computations were performed on a machine with 4 Dual Core AMD Opteron 64 Processor 8220 (2800 MHz) (only one processor could be used at a time) equipped with 32 GB RAM (at most 16 GB were allowed to us) running openSUSE 11 Linux.

We have used Risa/Asir version 20071022, Macaulay2 version 1.1 with version 1.0 of `Dmodules.m2` and Singular 3-1-0 with `bfun.lib` version 1.13.

We measure the total running time of each call to a system in a batch mode. In this time the initialization of the respective system, the loading of the respective example file, the actual computation and the writing of the output are included.

	Asir		Macaulay 2	Singular	
Example	`bfct`	`bfunction`	`globalBFunction`	`bfct`	`bfctAnn`
uw1	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
uw2	10:50	0:03	44:10	0:04	**0:02**
uw3	0:04	0:02	0:19	0:02	**0:01**
uw4	1h:11:16	**0:02**	17:56	0:04	**0:02**
uw5	**0:01**	0:02	**0:01**	**0:01**	**0:01**
uw6	37h:23:04$^\times$	0:30	5h:03:59$^\times$	0:28	**0:15**
uw7	5:29	0:07	5h:01:18$^\times$	0:08	**0:04**
uw8	5h:10:40$^\times$	**0:26**	7h:00:39$^\times$	0:39	1:34
uw9	7h:01:18$^\times$	**0:20**	5h:00:19$^\times$	0:47	4:22
uw10	23h:29:48	**0:06**	20h:16:16$^\times$	0:14	0:27
uw11	2h:52:34	**0:11**	8h:22:10$^\times$	0:22	1:20
uw12	16:57	**0:07**	12h:30:55$^\times$	0:19	1:31
uw13	41h:14:57$^\times$	**0:15**	27h:59:46$^\times$	0:22	4:39
uw14	0:04	0:07	0:03	0:14	**0:01**
uw15	61h:05:31$^\times$	15:42	26h:13:12$^\times$	5:37	**2:50**
uw16	67h:27:57$^\times$	6:06	3h:05:37$^\times$	2:03	**0:59**
uw17	48h:00:49$^\times$	**3:58**	3h:01:26$^\times$	5:59	1h:53:27$^\times$
uw18	29h:35:54$^\times$	7:26	4h:08:16$^\times$	**6:23**	12:24
uw19	24h:23:46$^\times$	**1:16**	3h:20:54$^\times$	1:31	10:10
uw20	6h:48:27	0:11	1h:43:02$^\times$	0:11	**0:06**
uw21	26h:34:58$^\times$	**2:53**	3h:34:07$^\times$	3:46	57:51
uw22	4h:04:05$^\times$	**2:13**	4h:01:43$^\times$	2:25	2h:00:36$^\times$
uw23	50h:07:44$^\times$	**2:39**	10h:07:53$^\times$	26:52	19h:20:52$^\times$
uw24	5h:08:07$^\times$	**4:03**	3h:14:29$^\times$	28:14	4h:42:23$^\times$
uw25	27h:56:42$^\times$	**2:32**	3h:09:18$^\times$	8:09	2h:11:52$^\times$
uw26	11h:48:32$^\times$	**4:55**	1h:16:51$^\times$	6:13	1h:22:20
uw27	3h:05:14$^\times$	**2:42**	11h:45:18$^\times$	4:46	11h:45:53$^\times$
uw28	10h:23:40$^\times$	**1:36**	3h:03:00$^\times$	3:12	2h:58:51$^\times$
uw29	3h:51:14$^\times$	**1:49**	10h:23:42$^\times$	2:55	2h:09:27$^\times$
uw30	5h:14:18$^\times$	**1:59**	3h:06:57$^\times$	3:11	2h:38:38$^\times$
uw31	5h:31:15$^\times$	**1:49**	5h:14:23$^\times$	3:09	2h:38:00$^\times$
uw32	10h:08:12$^\times$	**1:33**	5h:30:42$^\times$	2:38	8h:48:18$^\times$
uw33	5h:06:51$^\times$	**1:46**	10h:08:47$^\times$	7:18	2h:21:17$^\times$

Table 6.5: Hyperplane arrangements: Comparisons between Asir, Macaulay 2 and Singular

	Asir		Macaulay 2	Singular	
Example	`bfct`	`bfunction`	`globalBFunction`	`bfct`	`bfctAnn`
ab23	0:17	0:24	0:27	0:18	**0:04**
chal2	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
chal3	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
chal3b	**0:01**	**0:01**	0:05	0:20	**0:01**
cnu3	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
cnu4	0:02	0:04	0:03	**0:01**	**0:01**
cnu5	**0:01**	**0:01**	0:15	**0:01**	**0:01**
cnu6	0:54	1:39	14:01	**0:01**	**0:01**
cnu7s1	4:46	7:31	4h:03:39$^\times$	**0:06**	0:19
cusp23cusp32	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
cusp34	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
reiffen45	0:02	0:02	1:03	0:04	**0:01**
reiffen46	0:02	**0:01**	0:30	0:03	**0:01**
reiffen47	**0:02**	0:03	7:50	0:08	**0:02**
reiffen48	0:03	**0:01**	0:06	0:03	**0:01**
reiffen49	0:05	0:06	1h:33:18	0:22	**0:04**
reiffen56	0:48	0:10	n/a	0:35	**0:06**
reiffen57	0:10	**0:08**	n/a	0:45	**0:08**
reiffen58	0:11	**0:06**	n/a	0:55	0:12
reiffen59	0:15	**0:11**	n/a	1:40	0:20
reiffen66	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
reiffen67	1:15	1:04	n/a	4:50	**0:47**
reiffen68	0:54	0:15	n/a	2:35	**0:13**
reiffen69	0:35	0:14	n/a	0:46	**0:06**
reiffen77	0:02	**0:01**	**0:01**	**0:01**	**0:01**
reiffen78	15:54	**3:29**	n/a	24:54	4:44
reiffen79	7:24	**3:44**	n/a	56:31	5:46
reiffen88	0:04	**0:01**	0:03	0:02	**0:01**
reiffen89	24h:00:01$^\times$	**15:40**	7h:04:25$^\times$	2h:04:54	26:40
reiffen99	0:09	**0:01**	0:34	0:05	**0:01**
reiffen11	0:48	**0:03**	0:34	0:16	**0:03**
tt32	**0:01**	**0:01**	**0:01**	**0:01**	**0:01**
tt42	**0:01**	**0:01**	**0:01**	0:02	**0:01**
tt43	0:05	0:07	0:05	0:17	**0:01**
xyzcusp45	**1:09**	1:53	4h:18:35	3:08	2:52

Table 6.6: Various polynomials: Comparisons between Asir, Macaulay 2 and Singular

6.2.3 Ordering for the initial ideal based method

We request the computation of Bernstein-Sato polynomials via the initial ideal based method. We use the heuristically fast degree reverse lexicographical ordering $\prec_{\text{degrevlex}}$ (see Example 1.13) such that

$$x_1 \succ_{\text{degrevlex}} \dots \succ_{\text{degrevlex}} x_n \succ_{\text{degrevlex}} \partial_1 \succ_{\text{degrevlex}} \dots \succ_{\text{degrevlex}} \partial_n$$

and for a second computation $\prec_{\text{degrevlex}}$ with a permuted order of the variables determined by the valuation function `valvars` from `presolve.lib` [Gre10] according to Remark 4.4.

Example	`bfct` with $\prec_{\text{degrevlex}}$	`bfct` with heuristic ordering
ab23	**0:17**	0:18
chal2	**0:01**	**0:01**
chal3b	**0:20**	**0:20**
chal3	**0:01**	**0:01**
cnu3	0:02	**0:01**
cnu4	0:14	**0:01**
cnu5	1:16	**0:01**
cnu6	**0:01**	**0:01**
cnu7s1	23:09	**0:06**
cusp23cusp32	**0:01**	**0:01**
cusp34	**0:01**	**0:01**
reiffen11	**0:10**	0:16
reiffen66	**0:01**	**0:01**
reiffen77	**0:01**	**0:01**
reiffen88	**0:02**	**0:02**
reiffen89	**2h:04:36**	2h:04:54
reiffen99	**0:03**	0:05
tt32	**0:01**	**0:01**
tt42	**0:02**	**0:02**
tt43	**0:17**	**0:17**
xyzcusp45	45:41	**3:08**

Table 6.7: Various polynomials: Computation of Bernstein-Sato polynomials with and without an ordering as in Remark 4.4

Example	`bfct` with $\prec_{\text{degrevlex}}$	`bfct` with heuristic ordering
uw1	**0:01**	**0:01**
uw2	0:05	**0:04**
uw3	0:03	**0:02**
uw4	**0:03**	0:04
uw5	0:04	**0:01**
uw6	1:01	**0:28**
uw7	0:12	**0:08**
uw8	0:44	**0:39**
uw9	**0:37**	0:47
uw10	**0:08**	0:14
uw11	**0:19**	0:22
uw12	**0:15**	0:19
uw13	**0:19**	0:22
uw14	0:26	**0:14**
uw15	27:34	**5:37**
uw16	11:08	**2:03**
uw17	8:36	**5:59**
uw18	9:48	**6:23**
uw19	1:56	**1:31**
uw20	0:28	**0:11**
uw21	3:47	**3:46**
uw22	2:53	**2:25**
uw23	**11:21**	26:52
uw24	**19:28**	28:14
uw25	**7:49**	8:09
uw26	9:49	**6:13**
uw27	4:55	**4:46**
uw28	**1:42**	3:12
uw29	**2:16**	2:55
uw30	**2:54**	3:11
uw31	**2:52**	3:09
uw32	**2:24**	2:38
uw33	**5:56**	7:18

Table 6.8: Hyperplane arrangements: Computation of Bernstein-Sato polynomials with and without an ordering as in Remark 4.4

6.2.4 Syzygy-driven computation of the annihilator

We request the computation of Bernstein-Sato polynomials using the annihilator based approach with Algorithm 4.20 and with the syzygy-driven method from Algorithm 4.22 as described in Remark 4.21.
We have decided to perform this experiment only for the class of non-hyperplane arrangements since the annihilator based approach seems to be less efficient than the initial ideal based one for hyperplane arrangements.

Example	`bfctAnn` without computing syzygies	`bfctAnn` with computing syzygies
ab23	0:07	**0:04**
chal2	**0:01**	**0:01**
chal3	**0:01**	**0:01**
chal3b	**0:01**	**0:01**
cnu3	**0:01**	**0:01**
cnu4	**0:01**	**0:01**
cnu5	0:03	**0:01**
cnu6	0:04	**0:01**
cnu7s1	0:42	**0:19**
cusp23cusp32	**0:01**	**0:01**
cusp34	**0:01**	**0:01**
reiffen11	0:11	**0:03**
reiffen66	**0:01**	**0:01**
reiffen77	**0:01**	**0:01**
reiffen88	**0:01**	**0:01**
reiffen89	4h:07:59$^{\times}$	**26:40**
reiffen99	0:03	**0:01**
tt32	**0:01**	**0:01**
tt42	**0:01**	**0:01**
tt43	0:04	**0:01**
xyzcusp45	3:19	**2:52**

Table 6.9: Various polynomials: Computation of Bernstein-Sato polynomials with and without the syzygy-driven approach from Remark 4.21

6.2.5 Normal form computations

We request the computation of $\mathrm{NF}((t\partial_t)^i, \mathrm{in}_{(-w,w)}(I_f))$ for $1 \leq i \leq \deg(b_f(s))$, where $\mathrm{in}_{(-w,w)}(I_f)$ and $\deg(b_f(s))$ are given, i. e. these data were previously computed and are read in from prepared files.

We perform two separate computations for each example.

A "plain" one , i. e. without any "computational tricks":

```
ideal J = 1;
for (int i=1; i<=d; i++)              // d denotes deg(b_f(s))
{
  J[i+1] = NF(s^i,I);                 // I and s denote in_(-w,w)(I_f) and t∂_t
}
```

and another one making use of the previously computed normal forms according to Corollary 3.15:

```
poly p;
ideal J = 1, NF(s,I);
for (int i=2; i<=d; i++)              // d denotes deg(b_f(s))
{
  p = bracket(s^i-J[i],J[2]) + J[i]*J[2];
  J[i+1] = NF(p,I);                   // I and s denote in_(-w,w)(I_f) and t∂_t
}
```

Note that we have used a different machine for this experiment and for the one in the next section, namely a machine with four Intel core i7940 (2933 MHz) (only one processor could be used at a time) equipped with 12 GB RAM running Ubuntu 8.04.4 Linux.

Example	$\deg(b_f(s))$	$\mathrm{NF}(s^i, \mathrm{in}_{(-w,w)}(I_f)), i = 1, \ldots, \deg(b_f(s))$ standard	using Corollary 3.15
uw1	7	**0:01**	**0:01**
uw2	10	0:05	**0:01**
uw3	9	**0:01**	**0:01**
uw4	8	0:02	**0:01**
uw5	9	**0:01**	**0:01**
uw6	14	1:08	**0:10**
uw7	15	0:12	**0:02**
uw8	12	1:53	**0:21**
uw9	12	1:10	**0:15**
uw10	12	0:10	**0:04**
uw11	11	0:19	**0:05**
uw12	11	0:14	**0:04**
uw13	10	0:37	**0:15**
uw14	11	**0:01**	**0:01**
uw15	18	1h:27:25	**4:52**
uw16	19	11:47	**0:48**
uw17	16	1h:17:17	**6:01**
uw18	18	1h:24:52	**3:53**
uw19	18	5:12	**0:30**
uw20	15	0:10	**0:03**
uw21	17	27:02	**1:47**
uw22	17	11:56	**0:55**
uw23	14	3h:23:51	**34:50**
uw24	14	3h:56:23	**36:56**
uw25	14	1h:48:24	**14:00**
uw26	14	35:46	**5:18**
uw27	14	10:07	**1:52**
uw28	13	2:34	**0:47**
uw29	13	19:13	**2:55**
uw30	13	3:21	**0:50**
uw31	13	3:27	**0:51**
uw32	13	1:47	**0:37**
uw33	12	1h:16:31	**33:11**

Table 6.10: Computation of normal forms

6.2.6 Ordering and engine for the annihilator based method

We consider the Reiffen curve with parameters $(4, 5)$, i. e. $f = x^4 + y^5 + xy^4 \in \mathbb{K}[x, y]$, and measure the time for the computation of a Gröbner basis G of $\mathrm{Ann}_{D_2[s]}(f^s) + \langle f \rangle$ for previously computed $\mathrm{Ann}_{D_2[s]}(f^s)$ with respect to different orderings. We write G to a file and end the session.
Since $0 \neq b_f(s) \in \mathrm{Ann}_{D_2[s]}(f^s) + \langle f \rangle$, the intersection $(\mathrm{Ann}_{D_2[s]}(f^s) + \langle f \rangle) \cap \mathbb{K}[s]$ is nontrivial (cf. Corollary 4.16(a) and Lemma 4.15). Hence there exists an element g in the computed Gröbner basis which satisfies $\mathrm{lm}(g) = s^k$ for some $k \in \mathbb{N}$ (Lemma 3.7). We import G from the previously created file and print $|G|$. We also request to find k and print the size of g, i. e. the number of terms of g.
Finally, we start a third session, read in G again and call `pIntersect(s,G)` (Algorithm 3.12).
We perform each of these steps twice, one at a time for both Gröbner basis engines of Singular, `std` and `slimgb` [Bri06].
Table 6.11 shows the results of the computations. Note that the orderings used are given in the format of Singular. We refer again to the manual for detailed information. We also list hundredths of seconds.
Table 6.12 is a subtable of Table 6.11 consisting of the results for anti-elimination orderings for s and the Gröbner basis engine `slimgb`.

Ordering	std: Time for GB	std: Size of GB	std: $g \in GB : lm(g) = s^k$ k	std: $g \in GB : lm(g) = s^k$ Size	std: Time for pIntersect	slimgb: Time for GB	slimgb: Size of GB	slimgb: $g \in GB : lm(g) = s^k$ k	slimgb: $g \in GB : lm(g) = s^k$ Size	slimgb: Time for pIntersect
(s,x,y,Dx,Dy),(a(1,1,1,0,0),dp)	15:53.35	28	5	974	0:04.73	0:01.12	28	5	234	0:02.91
(s,x,y,Dx,Dy),(a(1,0,0,1,1),dp)	8:54.89†					0:15.32	45	6	136	0:01.50
(s,x,y,Dx,Dy),(a(0,1,1,1,1),dp)	15h:22:40$^\times$					0:18.78	54	13	14	0:00.37
(s,x,y,Dx,Dy),(a(1,0,0,0,0),dp)	0:00.26	8	2	47	0:03.94	0:00.04	8	2	76	0:03.83
(s,x,y,Dx,Dy),(a(0,1,1,0,0),dp)	1h:35:59†					1h:09:25$^\times$				
(s,x,y,Dx,Dy),(a(0,0,0,1,1),dp)	47:13.52†					0:13.46	52	13	14	0:00.34
(s,x,y,Dx,Dy),(dp(3),dp(2))	13:10.42	13	5	681	0:05.50	0:00.25	13	5	221	0:03.82
(s,x,y,Dx,Dy),(dp(1),dp(2),dp(2))	0:00.28	8	2	47	0:02.17	0:00.04	8	2	47	0:02.11
(s,x,y,Dx,Dy),(dp(1),dp(4))	0:00.27	8	2	47	0:02.04	0:00.04	8	2	47	0:02.03
(s,x,y,Dx,Dy),dp	16:21.05†					0:10.74	45	6	136	0:01.50
(s,x,y,Dx,Dy),wp(100,100,100,1,1)	1:33.10	28	5	228	0:02.31	0:01.47	28	5	315	0:02.40
(s,x,y,Dx,Dy),wp(100,1,1,100,100)	12:50.82†					0:13.96	45	6	136	0:01.30
(s,x,y,Dx,Dy),wp(1,100,100,100,100)	10h:01:26$^\times$					9:53.11	54	13	14	0:00.58
(s,x,y,Dx,Dy),wp(100,1,1,1,1)	0:00.16	8	2	70	0:01.94	0:00.04	8	2	77	0:01.97
(s,x,y,Dx,Dy),wp(1,1,1,100,100)	12:59.16†					0:20.60	52	13	14	0:00.34
(s,x,y,Dx,Dy),wp(1,100,100,1,1)	8h:28:36†					1h:10:23$^\times$				
(x,y,Dx,Dy,s),(a(1,1,1,1,0),dp)	5h:21:38$^\times$					0:18.02	54	13	14	0:00.38
(x,y,Dx,Dy,s),(a(1,1,0,0,1),dp)	1h:34:25†					0:01.65	29	5	320	0:04.35
(x,y,Dx,Dy,s),(a(0,0,1,1,1),dp)	8:25.20†					0:01.14	44	6	103	0:00.99
(x,y,Dx,Dy,s),(a(1,1,0,0,0),dp)	1h:03:13†					1h:10:51$^\times$				
(x,y,Dx,Dy,s),(a(0,0,0,0,1),dp)	0:00.28	8	2	47	0:03.93	0:00.04	8	2	76	0:03.96
(x,y,Dx,Dy,s),(a(0,0,1,1,0),dp)	1h:31:41†					0:16.72	52	13	14	0:00.33
(x,y,Dx,Dy,s),(dp(4),dp(1))	3h:19:32$^\times$					0:03.03	38	13	14	0:00.29
(x,y,Dx,Dy,s),(dp(2),dp(2),dp(1))	5h:41:02$^\times$					1h:24:58$^\times$				
(x,y,Dx,Dy,s),(dp(2),dp(3))	54:49.86†					1h:00:15$^\times$				
(x,y,Dx,Dy,s),dp	7:03.83†					0:02.25	55	7	102	0:00.91
(x,y,Dx,Dy,s),wp(100,100,100,100,1)	1h:03:28$^\times$					9:54.46	54	13	14	0:00.59
(x,y,Dx,Dy,s),wp(100,100,1,1,100)	16:40.86†					0:02.29	29	5	397	0:04.23
(x,y,Dx,Dy,s),wp(1,1,100,100,100)	2:05.60	44	6	103	0:00.87	0:01.21	44	6	103	0:00.87
(x,y,Dx,Dy,s),wp(1,1,100,100,1)	12:54.70†					0:17.22	52	13	14	0:00.33
(x,y,Dx,Dy,s),wp(100,100,1,1,1)	1h:04:24$^\times$					1h:03:58$^\times$				
(x,y,Dx,Dy,s),wp(1,1,1,1,100)	0:00.18	8	2	70	0:01.93	0:00.05	8	2	77	0:01.94
(x,y,s,Dx,Dy),(a(1,1,1,0,0),dp)	45:14.49†					0:01.50	28	5	320	0:03.05
(x,y,s,Dx,Dy),(a(1,1,0,1,1),dp)	1h:00:59$^\times$					0:20.62	54	13	14	0:00.37
(x,y,s,Dx,Dy),(a(0,0,1,1,1),dp)	9:14.08†					0:15.86	45	6	136	0:01.48
(x,y,s,Dx,Dy),(a(1,1,0,0,0),dp)	1h:23:35$^\times$					1h:14:14$^\times$				
(x,y,s,Dx,Dy),(a(0,0,0,1,1),dp)	1h:12:44$^\times$					0:18.10	52	13	14	0:00.32
(x,y,s,Dx,Dy),(a(0,0,1,0,0),dp)	0:00.28	8	2	47	0:03.88	0:00.05	8	2	76	0:03.86
(x,y,s,Dx,Dy),(dp(2),dp(1),dp(2))	1h:05:25$^\times$					1h:02:28$^\times$				
(x,y,s,Dx,Dy),(dp(3),dp(2))	1h:00:15$^\times$					0:32.74	30	5	364	52:05.44

Table 6.11: Computation of $\mathrm{Ann}_{D_2}(f^s) + \langle f \rangle$ (continued on next page)

Ordering	std: Time for GB	std: Size of GB	std: $g \in GB : lm(g) = s^k$ k	std: $g \in GB : lm(g) = s^k$ Size	std: Time for pIntersect	slimgb: Time for GB	slimgb: Size of GB	slimgb: $g \in GB : lm(g) = s^k$ k	slimgb: $g \in GB : lm(g) = s^k$ Size	slimgb: Time for pIntersect
(x,y,s,Dx,Dy),(dp(2),dp(3))	1h:04:59$^{\times}$					1h:15:38$^{\times}$				
(x,y,s,Dx,Dy),dp	13:06.19†					0:12.66	47	6	135	0:01.47
(x,y,s,Dx,Dy),wp(100,100,100,1,1)	2:36.20	28	5	228	0:02.45	0:02.04	28	5	397	0:02.82
(x,y,s,Dx,Dy),wp(100,100,1,100,100)	1h:06:23$^{\times}$					9:53.50	54	13	14	0:00.58
(x,y,s,Dx,Dy),wp(1,1,100,100,100)	14:09.98†					0:15.41	45	6	136	0:01.27
(x,y,s,Dx,Dy),wp(100,100,1,1,1)	1h:17:27$^{\times}$					1h:33:20$^{\times}$				
(x,y,s,Dx,Dy),wp(1,1,1,100,100)	14:01.65†					0:19.41	52	13	14	0:00.33
(x,y,s,Dx,Dy),wp(1,1,100,1,1)	0:00.18	8	2	70	0:01.91	0:00.04	8	2	77	0:01.93
(s,Dx,Dy,x,y),(a(1,1,1,0,0),dp)	7:25.64†					0:05.04	49	6	149	0:01.40
(s,Dx,Dy,x,y),(a(1,0,0,1,1),dp)	22:54.97†					0:02.90	33	5	234	0:02.81
(s,Dx,Dy,x,y),(a(0,1,1,1,1),dp)	1h:10:10$^{\times}$					0:17.57	56	13	14	0:00.39
(s,Dx,Dy,x,y),(a(1,0,0,0,0),dp)	0:00.23	8	2	63	0:03.64	0:00.05	8	2	83	0:03.55
(s,Dx,Dy,x,y),(a(0,1,1,0,0),dp)	52:07.96†					0:21.39	53	13	14	0:00.39
(s,Dx,Dy,x,y),(a(0,0,0,1,1),dp)	1h:03:43$^{\times}$					2h:18:49$^{\times}$				
(s,Dx,Dy,x,y),(dp(3),dp(2))	0:00.22	8	2	63	0:02.10	0:00.04	8	2	63	0:02.14
(s,Dx,Dy,x,y),(dp(1),dp(2),dp(2))	0:00.25	8	2	63	0:01.83	0:00.04	8	2	63	0:01.81
(s,Dx,Dy,x,y),(dp(1),dp(4))	0:00.23	8	2	63	0:01.74	0:00.04	8	2	63	0:01.75
(s,Dx,Dy,x,y),dp	22:53.15†					0:04.39	49	6	149	0:01.39
(s,Dx,Dy,x,y),wp(100,100,100,1,1)	27:08.61	49	6	149	0:01.21	0:04.81	49	6	149	0:01.20
(s,Dx,Dy,x,y),wp(100,1,1,100,100)	4:45.06	33	5	228	0:02.24	0:03.94	33	5	310	0:02.27
(s,Dx,Dy,x,y),wp(1,100,100,100,100)	26:24.66†					1h:03:37$^{\times}$				
(s,Dx,Dy,x,y),wp(100,1,1,1,1)	0:00.13	8	2	77	0:01.71	0:00.04	8	2	84	0:01.72
(s,Dx,Dy,x,y),wp(1,1,1,100,100)	1h:04:42$^{\times}$					1h:08:51$^{\times}$				
(s,Dx,Dy,x,y),wp(1,100,100,1,1)	14:34.25†					0:12.70	53	13	14	0:00.36
(Dx,Dy,x,y,s),(a(1,1,1,1,0),dp)	1h:01:14$^{\times}$					0:16.74	56	13	14	0:00.39
(Dx,Dy,x,y,s),(a(1,1,0,0,1),dp)	4:33.85	44	6	437	0:01.62	0:00.99	44	6	105	0:01.03
(Dx,Dy,x,y,s),(a(0,0,1,1,1),dp)	1h:11:18†					0:02.93	33	5	315	0:03.64
(Dx,Dy,x,y,s),(a(1,1,0,0,0),dp)	1h:38:03†					0:14.58	54	13	14	0:00.38
(Dx,Dy,x,y,s),(a(0,0,0,0,1),dp)	0:00.24	8	2	63	0:03.55	0:00.04	8	2	83	0:03.53
(Dx,Dy,x,y,s),(a(0,0,1,1,0),dp)	1h:00:27†					1h:10:45$^{\times}$				
(Dx,Dy,x,y,s),(dp(4),dp(1))	4h:41:08$^{\times}$					0:02.97	44	13	14	0:00.24
(Dx,Dy,x,y,s),(dp(2),dp(2),dp(1))	1h:03:43$^{\times}$					0:03.44	38	13	14	0:00.25
(Dx,Dy,x,y,s),(dp(2),dp(3))	38:24.37†					0:01.65	41	13	14	0:00.21
(Dx,Dy,x,y,s),dp	5:40.74†					0:02.49	57	7	102	0:00.93
(Dx,Dy,x,y,s),wp(100,100,100,100,1)	24:54.35†					1h:06:53	56	13	14	0:04.47
(Dx,Dy,x,y,s),wp(100,100,1,1,100)	1:55.22	44	6	105	0:00.86	0:00.93	44	6	105	0:00.87
(Dx,Dy,x,y,s),wp(1,1,100,100,100)	15:51.17†					0:03.63	33	5	386	0:03.18
(Dx,Dy,x,y,s),wp(1,1,100,100,1)	1h:07:28$^{\times}$					1h:21:54$^{\times}$				
(Dx,Dy,x,y,s),wp(100,100,1,1,1)	13:37.41†					0:20.03	54	13	14	0:00.39
(Dx,Dy,x,y,s),wp(1,1,1,1,100)	0:00.13	8	2	77	0:01.70	0:00.05	8	2	84	0:01.71

Table 6.11: Computation of $\mathrm{Ann}_{D_2}(f^s) + \langle f \rangle$ (continued on next page)

Ordering	std Time for GB	std Size of GB	std $g \in GB : lm(g) = s^k$ k	std $g \in GB : lm(g) = s^k$ Size	std Time for pIntersect	slimgb Time for GB	slimgb Size of GB	slimgb $g \in GB : lm(g) = s^k$ k	slimgb $g \in GB : lm(g) = s^k$ Size	slimgb Time for pIntersect
(Dx,Dy,s,x,y),(a(1,1,1,0,0),dp)	5:48.88†					0:00.94	44	6	105	0:01.01
(Dx,Dy,s,x,y),(a(1,1,0,1,1),dp)	1h:00:12$^\times$					0:15.88	56	13	14	0:00.40
(Dx,Dy,s,x,y),(a(0,0,1,1,1),dp)	22:44.43†					0:02.14	33	5	234	0:02.77
(Dx,Dy,s,x,y),(a(1,1,0,0,0),dp)	51:54.11†					0:19.32	53	13	14	0:00.40
(Dx,Dy,s,x,y),(a(0,0,0,1,1),dp)	1h:59:23$^\times$					1h:11:55$^\times$				
(Dx,Dy,s,x,y),(a(0,0,1,0,0),dp)	0:00.24	8	2	63	0:03.58	0:00.05	8	2	83	0:03.50
(Dx,Dy,s,x,y),(dp(2),dp(1),dp(2))	14:49.13†					0:05.57	44	13	141	0:00.33
(Dx,Dy,s,x,y),(dp(3),dp(2))	2:07.19	43	6	604	0:01.25	0:01.03	43	6	334	0:01.07
(Dx,Dy,s,x,y),(dp(2),dp(3))	21:06.61†					0:01.64	44	13	14	0:00.21
(Dx,Dy,s,x,y),dp	1:51.67	44	6	105	0:01.02	0:00.81	44	6	105	0:01.02
(Dx,Dy,s,x,y),wp(100,100,100,1,1)	2:06.40	44	6	105	0:00.90	0:00.92	44	6	105	0:00.89
(Dx,Dy,s,x,y),wp(100,100,1,100,100)	24:07.79†					1h:08:23	56	13	14	0:04.45
(Dx,Dy,s,x,y),wp(1,1,100,100,100)	4:43.86	33	5	228	0:02.17	0:03.18	33	5	310	0:02.23
(Dx,Dy,s,x,y),wp(100,100,1,1,1)	12:27.79†					0:10.51	53	13	14	0:00.36
(Dx,Dy,s,x,y),wp(1,1,1,100,100)	1h:06:58$^\times$					1h:40:22$^\times$				
(Dx,Dy,s,x,y),wp(1,1,100,1,1)	0:00.13	8	2	77	0:01.71	0:00.04	8	2	84	0:01.73

Table 6.11: Computation of $\mathrm{Ann}_{D_2}(f^s) + \langle f \rangle$

Ordering	slimgb **Time for GB**	**Size of GB**	$\mathbf{g \in GB : \mathrm{lm}(g) = s^k}$ **k**	**Size**	**Time for** `pIntersect`
`(s,x,y,Dx,Dy),(a(0,1,1,1,1),dp)`	**0:18.78**	54	13	14	0:00.37
`(s,x,y,Dx,Dy),wp(1,100,100,100,100)`	**9:53.11**	54	13	14	0:00.58
`(x,y,Dx,Dy,s),(a(1,1,1,1,0),dp)`	**0:18.02**	54	13	14	0:00.38
`(x,y,Dx,Dy,s),(dp(4),dp(1))`	**0:03.03**	38	13	14	0:00.29
`(x,y,Dx,Dy,s),wp(100,100,100,100,1)`	**9:54.46**	54	13	14	0:00.59
`(x,y,s,Dx,Dy),(a(1,1,0,1,1),dp)`	**0:20.62**	54	13	14	0:00.37
`(x,y,s,Dx,Dy),wp(100,100,1,100,100)`	**9:53.50**	54	13	14	0:00.58
`(s,Dx,Dy,x,y),(a(0,1,1,1,1),dp)`	**0:17.57**	56	13	14	0:00.39
`(Dx,Dy,x,y,s),(a(1,1,1,1,0),dp)`	**0:16.74**	56	13	14	0:00.39
`(Dx,Dy,x,y,s),(dp(4),dp(1))`	**0:02.97**	44	13	14	0:00.24
`(Dx,Dy,x,y,s),(dp(2),dp(2),dp(1))`	**0:03.44**	38	13	14	0:00.25
`(Dx,Dy,x,y,s),wp(100,100,100,100,1)`	**1h:06:53**	56	13	14	0:04.47
`(Dx,Dy,s,x,y),(a(1,1,0,1,1),dp)`	**0:15.88**	56	13	14	0:00.40
`(Dx,Dy,s,x,y),wp(100,100,1,100,100)`	**1h:08:23**	56	13	14	0:04.45

Table 6.12: Subtable of Table 6.11 with anti-elimination orderings for s and `slimgb`

6.2.7 Conclusion

Tables 6.5 and 6.6 show that both Asir and Singular are superior to Macaulay 2, while none of the first two systems is distinctly better than the other one.
We would like to stress that the implementation in Asir incorporates modular methods for Gröbner basis computations as well as for the computation of intersections. We have deliberately decided not to use any modular techniques in order to see what we can achieve by solely computing in characteristic zero. Nevertheless, our implementation is designed in such a way that integrating other methods and approaches, respectively, can easily be done.
In addition, the data also suggest that neither `bfct` nor `bfctAnn` is clearly superior to the other algorithm. The initial ideal based method performs better on hyperplane arrangements while the annihilator based approach seems to be more efficient for other kinds of input.
The results in Tables 6.7 and 6.8 suggest that there are examples where the heuristic ordering for the computation of the initial ideal might be twice as slow compared to the degree reverse lexicographical one. Nevertheless, we see that it is preferable to use it especially on non-hyperplane arrangements.
It is desirable to use the syzygy-driven computation of the s-parametric annihilator for non-hyperplane arrangements (Table 6.9). Note that the computation of the syzygy module takes place in the commutative ring $\mathbb{K}[x_1, \ldots, x_n]$ and is not hard in the examples we studied.
The computation of normal forms by making use of previously computed ones following Corollary 3.15 is clearly better, even on "relatively easy" examples as seen in Table 6.10.
Table 6.11 shows that it is evident that `slimgb` is clearly superior to `std` for Gröbner basis computations in $D_n[s]$. Moreover, we can conclude that anti-elimination orderings for s are much better suited in order to achieve an efficient computation. Moreover, *block orderings* (see [GP08]) are preferable over weighted degree ones (Table 6.12).

7 Conclusion and future work

We have seen that there are two distinct methods for the computation of Bernstein-Sato polynomials. One is based on the s-parametric annihilator while the other one is based on the initial ideal, being a special case of the concept of b-functions for ideals.

We have also seen that in practice, none of these approaches is clearly better than the other one in general, but for certain classes of input, there are differences in the performance.

One of the general difficulties in computations with D-modules is intermediate coefficient swell. In the algorithmic approach to the computation of b-functions, M. Noro suggested the use of *modular techniques* in the Weyl algebra to overcome this problem [Nor02]. A generalization of the existing theory on modular algorithms for the computation of Gröbner bases in commutative rings over $\mathbb{Q}$ [Arn03] to arbitrary G-algebras over $\mathbb{Q}$ along with an implementation is desirable.

Further, in the commutative case, the *Hilbert-driven Buchberger algorithm* [Tra96] can often seriously improve the efficiency of Gröbner basis computations. It is interesting to investigate whether, and if so, how this theory can be carried to the setting of arbitrary G-algebras. For that purpose, the theory of *Hilbert polynomials* and *Hilbert series* as well as algorithms have to be developed and understood in the non-commutative setting. In this context, also the use of *involutive bases* [Ape98, GB98, HSS02] should be examined. Due to recent results [Sei10], the use of certain *involutive division* allows the simultaneous computation of a Gröbner basis and the Hilbert polynomial.

These improvements would affect both approaches for the computation of Bernstein-Sato polynomials.

Another perspective concerns solely the initial ideal based method. While computing Gröbner bases in the weighted homogenized Weyl algebra, one would be able to keep the degree in the homogenization variable h small, and thus improve the efficiency of the computation, if one uses *saturation* in the sense of dividing a polynomial by the content in h. That is, given $p = \sum_{\alpha,\beta,\lambda} c_{\alpha\beta\lambda} x^\alpha \partial^\beta h^\lambda$, set $\mu := \min\{\lambda \mid c_{\alpha\beta\lambda} \neq 0\}$ and replace p by $\frac{p}{h^\mu}$. Since h will be replaced by 1 in the course of dehomogenization, no information would be lost.

In addition, we have investigated the problem of computing the intersection of an ideal with a subalgebra. It remains an open question if there is a way to determine whether this intersection is zero for both the case where the subalgebra is generated by a single element and the multivariate case, without computing the intersection by means of Gröbner basis elimination.

M. Noro also suggested the use of *modular techniques* for the computation of the intersection needed for the global b-function [Nor02]. A generalization of this approach to intersections of ideals with arbitrary multivariate $\mathbb{Q}$-algebras is desirable, both from a theoretical and an implementational point of view.
An examination of the multivariate intersection problem in the context of generalized Bernstein-Sato polynomials, Bernstein-Sato ideals as well as Bernstein-Sato polynomials for varieties is worthwhile, too.
Moreover, it sounds promising to utilize the concept of principal intersection in the context of certain localizations.

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Erklärung

Hiermit versichere ich, dass ich die Aufgabenstellung selbständig bearbeitet und keine außer den angegebenen Hilfsmitteln verwendet habe.

Aachen, den 14. Juli 2010

Daniel Andres