ON THE NUMBER OF CONSTITUENTS OF INDUCED MODULES OF ARIKI-KOIKE ALGEBRAS

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Abstract. We examine the crystal graph of the $\widehat{\mathfrak{sl}}_e$-module arising from an $\widehat{\mathfrak{sl}}_e$-categorification to study the defining endo-functors of the categorification. This yields lower bounds on the number of irreducible constituents of certain objects. We use Ariki's categorification result on Ariki-Koike algebras to obtain a new lower bound on the number of constituents of their parabolically induced modules. In particular this will imply reducibility of every induced module.

Introduction

Denote by $\widehat{\mathfrak{sl}}_e$ the affine Lie algebra of type $A^{(1)}_{e-1}$, by $U(\widehat{\mathfrak{sl}}_e)$ its enveloping algebra and by $U_u(\widehat{\mathfrak{sl}}_e)$ its quantum enveloping algebra. A large portion of the structure of certain classes of $U_u(\widehat{\mathfrak{sl}}_e)$- and $U(\widehat{\mathfrak{sl}}_e)$-modules can be encoded in so-called crystal graphs via the concepts of crystal bases and perfect bases, respectively, cf. e.g. [HK02, BK07]. These crystal graphs have a nice combinatorial description stemming from the realisation of irreducible highest weight modules as submodules of Fock spaces cf. [FLO+99]. We will exploit this combinatorial description to study categories possessing a so-called $\widehat{\mathfrak{sl}}_e$-categorification of a $\mathbb{C}$-linear abelian category C, as defined in [Rou08]. One key ingredient to such a categorification is a pair of adjoint endo-functors (U, V) of C which decompose as direct sums of e summands and this decomposition yields an $\widehat{\mathfrak{sl}}_e$-module structure on the complexification of the Grothendieck group of C, i.e. on $\mathbb{C} \otimes_{\mathbb{Z}} R_0(C)$.
If this $\widehat{\mathfrak{sl}}_e$-module is an element of the so-called category O_{int} of $\widehat{\mathfrak{sl}}_e$-modules, we combine results by Shan and Chuang-Rouquier, cf. [Sha11, CR08], with a combinatorial observation to obtain a new lower bound on the number of constituents of images under V.
These results are applicable to a number of settings, in particular to category O of rational Cherednik algebras and the representations of Ariki-Koike algebras, cf. [Sha11, Ari02]. Both are closely related to complex reflection groups of type $G(r, 1, n)$, that is, groups of the form $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group on n letters. In both cases, the functor V is given by parabolic induction, thus we can use our general result to study lower bounds on the number of constituents of parabolically induced modules.

Motivated by this result on parabolic induction for certain Ariki-Koike algebras we go on to prove analogous bounds for Ariki-Koike algebras with arbitray invertible parameters, as not all parameter choices are covered by the $\widehat{\mathfrak{sl}}_e$-categorification result: If K is a field, then the Ariki-Koike algebra over K is defined via generators and relations involving parameters $q, Q_1, \ldots, Q_r \in K^*$. The $\widehat{\mathfrak{sl}}_e$-categorification result is known to hold in the case that $q \neq 1$ is a root of unity of finite order and the parameters $Q_1, \ldots, Q_r$ are so-called

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q-connected, cf. [Ari02, Thm 12.5]. Hence, to obtain a complete result we first reduce the task of computing the number of constituents of induced modules to q-connected parameters via the Morita equivalence result by Dipper-Mathas, cf. [DM02].
Then it remains to consider the cases that q is either 1 or has infinite order in K^*. While the latter is handled quite similarly to our study of $\widehat{\mathfrak{sl}}_e$-categorification, the former requires some hands-on computation.
The main result on parabolic induction of Ariki-Koike algebras with arbitrary invertible parameters is then given in Theorem 2.22.

This article is structured as follows:
We establish the necessary vocabulary for the representation theory of $\widehat{\mathfrak{sl}}_e$, in particular integrable modules, category $\mathcal{O}_{\text{int}}$, perfect bases, and crystal graphs. Then we introduce certain crystal graphs, study their combinatorics and indicate how to obtain the crystal graphs of all elements of $\mathcal{O}_{\text{int}}$ from the ones we defined. After recalling the definition of $\widehat{\mathfrak{sl}}_e$-categorification we present our main result in Theorem 1.15.
As an application we consider parabolic induction in rational cyclotomic Cherednik algebras, cf. Theorem 1.18.

The second chapter is concerned with the study of Ariki-Koike algebras and their parabolic induction. To obtain the desired lower bound we reduce the task to the case of so-called q-connected parameter sets and handle the cases not covered by Ariki's $\widehat{\mathfrak{sl}}_n$-categorification result separately.
We close by proving an analogue of Theorem 2.22 for the closely related degenerate cyclotomic Hecke algebras, cf. Theorem 2.24.

1. $\widehat{\mathfrak{sl}}_e$-categorification and crystal graphs

1.1. **The Kac-Moody algebra and crystal graphs.** We start of by defining the affine Lie algebra $\widehat{\mathfrak{sl}}_e$, i.e. the Kac-Moody algebra of type A^1_{e-1} following [HK02]. Let $e \geq 2$ be an integer. We give the following definitions only for $e \geq 3$, but for $e = 2$ the construction is similiar, cf. [Kac90] for details.
Let $\mathfrak{h}$ be a $\mathbb{C}$-vector space with basis $\{h_1, \dots, h_{e-1}, d\}$ and $\{\Lambda_0, \dots, \Lambda_{e-1}, \partial\}$ a $\mathbb{C}$-basis of $\mathfrak{h}^*$ such that

$$\Lambda_i(h_j) = \delta_{i,j}, \quad \Lambda_i(d) = \partial(h_i) = 0, \quad \partial(d) = 1,$$

for $0 \leq i, j \leq e-1$. For ease of notation we set $\Lambda_z := \Lambda_{z \pmod e}$ for any integer z.
For $0 \leq i \leq e-1$ we define further elements of $\mathfrak{h}^*$ by

$$\alpha_i := -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} + \delta_{0,i}\partial.$$

The *affine Lie algebra* $\widehat{\mathfrak{sl}}_e$ is the Lie algebra generated by the elements e_i, f_i for $0 \leq i \leq e-1$ and $\{h_1, \dots, h_{e-1}, d\}$ subject to the following relations:

$$\begin{gathered}
[h, e_i] = \alpha_i(h)e_i, \\
[h, f_i] = -\alpha_i(h)f_i, \\
[e_i, f_j] = \delta_{i,j}h_i, \quad [h, h'] = 0, \\
[e_i, [e_i, e_j]] = [f_i, [f_i, f_j]] = 0, \text{ if } (i-j) \equiv \pm 1 \pmod e, \\
[e_i, e_j] = [f_i, f_j] = 0, \text{ if } (i-j) \not\equiv \pm 1 \pmod e,
\end{gathered}$$

for $h, h' \in \mathfrak{h}$ and $0 \leq i, j \leq e-1$.

We call the Λ_i the *fundamental weights* of $\widehat{\mathfrak{sl}}_e$ and ∂ the *null root*. Furthermore, the α_i are known as *simple roots* and the h_i as *simple co-roots* We define the *weight lattice*

$P := \mathbb{Z}\,\partial \oplus \bigoplus_{i=0}^{e-1} \mathbb{Z}\,\Lambda_i$ and the *dominant integral weights* $P^+ := \mathbb{Z}\,\partial \bigoplus_{i=0}^{e-1} \mathbb{Z}_{\geq 0}\,\Lambda_i$. Finally, we set $\overline{P^+} := \bigoplus_{i=0}^{e-1} \mathbb{Z}_{\geq 0}\,\Lambda_i$, the *classical dominant integral weights.*

In the following we will be concerned with certain representations of $\widehat{\mathfrak{sl}}_e$ or, equivalently, of $U(\widehat{\mathfrak{sl}}_e)$, its universal enveloping algebra. All modules studied here will have a *weight space decomposition*: For an $\widehat{\mathfrak{sl}}_e$-module M and some $\lambda \in \mathfrak{h}^*$ denote by $M_\lambda := \{m \in M \mid hm = \lambda(h)m \text{ for all } h \in \mathfrak{h}\}$ the *weight space of M of weight* λ.

An $\widehat{\mathfrak{sl}}_e$-representation is called *integrable* if the Chevalley generators e_i and f_i for $0 \leq i \leq e-1$ of $\widehat{\mathfrak{sl}}_e$ act locally nilpotently. We say that an $\widehat{\mathfrak{sl}}_e$-module M is in *category* O_{int} if

- M is integrable,
- M has a weight space decomposition $M = \oplus_\lambda M_\lambda$ and M_λ is finite dimensional for all λ,
- there exists a finite set $F \subseteq P$ such that $\mathrm{wt}(M) \subseteq F + \sum_{j=0}^{e-1} \mathbb{Z}_{\leq 0}\,\alpha_i$, where α_i is the i'th simple root of $\widehat{\mathfrak{sl}}_e$ and $\mathrm{wt}(M)$ is the set of weights λ in P such that $M_\lambda \neq 0$.

If M is in O_{int}, then M decomposes as a direct sum of *irreducible highest weight modules* $L(\lambda)$ with weight λ, where λ is in P^+, and every irreducible weight module $L(\lambda)$ with λ in P^+ is an element of O_{int}.
Every module M in O_{int} has a *perfect basis* in the sense of [BK07], i.e. a basis B consisting of weight vectors equipped with functions $\widetilde{E}_i, \widetilde{F}_i : B \to B\dot{\cup}\{0\}$ for $0 \leq i \leq e-1$ such that

- for b, b' in B it is $\widetilde{F}_i(b) = b'$ if and only if $\widetilde{E}_i(b') = b$,
- It is $\widetilde{E}_i(b) \neq 0$ if and only if $e_i b \neq 0$, where $e_0, \ldots, e_{n-1}$ and $f_0, \ldots, f_{n-1}$ are again the Chevalley generators of $\widehat{\mathfrak{sl}}_e$,
- if $e_i b \neq 0$, then
$$e_i b \in \mathbb{C}^*\,\widetilde{E}_i(b) + V_i^{<\ell_i(b)-1},$$
where $\ell_i(v) := \max\{j \geq 0 \mid e_i^j v \neq 0\}$ and $V_i^{<k} := \{v \in M \mid \ell_i(v) < k\}$.

To a perfect basis of M we can associate an abstract crystal in the sense of [HK02, Definition 4.5.1]. However, we will only be interested in its *crystal graph*. If M is in O_{int} with a crystal basis B, then the crystal graph associated to B is a directed graph with coloured edges, whose vertex set is B and for b, b' in B there is an edge $b \xrightarrow{i} b'$ with label i if and only if $\widetilde{F}_i(b) = b'$.

Definition 1.1. *A* crystal graph isomorphism *is an isomorphism of couloured graphs between crystal graphs of perfect bases, i.e. if B and C are perfect bases of modules M and N, then a crystal isomorphism is a bijection $\phi : B \to C$ such that there is an edge $b \xrightarrow{i} b'$ in the crystal graph associated to B if and only if there is an edge $\phi(b) \xrightarrow{i} \phi(b')$ in the crystal graph associated to C.*

For modules in O_{int} there is only one associated crystal graph:

Lemma 1.2. *If B and B′ are two perfect bases of $M \in O_{int}$, then the crystal graphs associated to B and B′ are isomorphic. Thus, it makes sense to speak of the crystal graph associated to M.*

Proof. This follows from [BK07, Main Thm 5.37] just as in the proof of [Sha11, Thm 6.3]. □

Via Fock space theory it is possible to determine the crystal graph associated to any module in O_{int}. This requires some groundwork:

For $r \in \mathbb{Z} \geq 0$ set $\widetilde{\mathbb{Z}^r_{\geq 0}} := \{(s_1, \dots, s_r) \mid 0 \leq s_1 \leq \cdots \leq s_r < e\}$ and for the time being fix some positive integer r and let $\mathbf{s} = (s_1, \dots, s_r) \in \widetilde{\mathbb{Z}^r_{\geq 0}}$.
Let n be a non-negative integer. A *partition of* n is a tuple $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of non-increasing non-negative integers $\alpha_1, \dots, \alpha_\ell$ such that $|\alpha| := \sum_{i=1}^{\ell} \alpha_i = n$. We write $\alpha \vdash n$ if α is a partition of n. An *r-multipartition* of n is a tuple $\lambda = (\mu^{(1)}, \dots, \lambda^{(r)})$ where each $\lambda^{(i)}$ is a partition and $|\lambda| := \sum_{i=1}^{r} |\lambda^{(i)}| = n$. We write $\lambda \vdash_r n$ if λ is an r-multipartition of n. The *Young diagram* $[\lambda]$ of an r-multipartition $\lambda \vdash_r n$ is the set

$$\left\{(a, b, c) \mid 1 \leq \lambda_a^{(c)} \leq b,\ 1 \leq c \leq r\right\}.$$

The elements of $[\lambda]$ are called *nodes*. More generally, we call any element of $\mathbb{N} \times \mathbb{N} \times \{1, \dots, r\}$ a node. A node x is called an *addable node of* λ if $x \notin [\lambda]$ and $[\lambda] \cup \{x\}$ is the Young diagram of an r-multipartition of $n + 1$. We write $\lambda \cup \{x\}$ for the corresponding r-multipartition.
Similarly, x is called a *removable node of* λ if $x \in [\lambda]$ and $[\lambda] \setminus \{x\}$ is the Young diagram of an r-multipartition of $n - 1$. We write $\lambda \setminus \{x\}$ for the corresponding r-multipartition.

Definition 1.3. *Let $\lambda \vdash_r n$. The* residue of a node $x = (a, b, c) \in [\lambda]$ *(with respect to* $\mathbf{s}$*) is defined as* $\mathrm{res}(x) := b - a + s_c \pmod e$. *If* $\mathrm{res}(x) \equiv i \pmod e$ *for* $0 \leq i \leq e - 1$*, then we call x an i-node.*
Addable i-nodes are called i-addable and removable i-nodes are called i-removable.

Definition 1.4. *For two nodes $x := (a, b, c)$ and $y := (a', b', c')$ we say that* x lies above or higher than y *if $c < c'$ or $c = c'$ and $a < a'$ or $c = c', a = a'$ and $b > b'$. We also say that y lies below or lower than x.*
This is the abstract notion of the usual visual way of writing down the diagram of λ by depicting the diagrams of the components below one another, starting with $\lambda^{(1)}$. This yields a total order on the Young diagram $[\lambda]$.

We follow [Ari02] to define a number of different objects to construct a certain crystal graph.
We define normal, co-normal, good, and co-good nodes of $\lambda \vdash_r n$:
Choose some $1 \leq i \leq e-1$ and write down the sequence of addable and removable i-nodes sorted from highest to lowest. Encode every addable node with the symbol $+_i$ and every removable one with the symbol $-_i$. The resulting sequence is called the *i-signature of* λ.
Now recursively remove all pairs $-_i+_i$ from this sequence until this is no longer possible to finally obtain the *reduced i-signature of* λ, which we denote by $w_{e,\mathbf{s}}(\lambda)$.
The nodes corresponding to $-_i$ in $w_{e,\mathbf{s}}(\lambda)$ are called i-normal.
The nodes corresponding to $+_i$ in $w_{e,\mathbf{s}}(\lambda)$ are called i-co-normal.
The highest i-normal node is called i-good.
The lowest i-co-normal node is called i-co-good. A node is called normal (co-normal, good, co-good) if it is i-normal (i-co-normal, i-good, i-co-good) for some i.
We use these operators to define some directed graph with coloured edges:

Definition 1.5. *Let $\mathcal{P}_{n,r}$ be the set of all r-multipartitions of n and set $\mathcal{P}_r := \cup_{n \in \mathbb{Z}_{\geq 0}} \mathcal{P}_{n,r}$. On $\mathcal{P}_r$ define operators $\widetilde{e_i}$ and $\widetilde{f_i}$ by*

$$\widetilde{e_i}(\lambda) = \begin{cases} \lambda \setminus \{x\}, & \text{if } x \text{ is the } i\text{-good node of } \lambda \\ 0, & \text{if } \lambda \text{ does not have an } i\text{-good node} \end{cases}$$

and

$$\widetilde{f_i}(\lambda) = \begin{cases} \lambda \cup \{x\}, & \text{if } x \text{ is the } i\text{-co-good node of } \lambda \\ 0, & \text{if } \lambda \text{ does not have an } i\text{-co-good node.} \end{cases}$$

By $B_e(\mathcal{F}_{\mathbf{s}})$ we denote the directed graph with vertex set $\mathcal{P}_r$ and edges $\lambda \xrightarrow{i} \mu$ if and only if $\widetilde{f_i}(\lambda) = \mu$, or, equivalently, $\widetilde{e_i}(\mu) = \lambda$.
Furthermore, we denote by $B_e(\mathbf{s})$ the subgraph of $B_e(\mathcal{F}_{\mathbf{s}})$ defined by the connected component containing the empty partition $\emptyset \vdash_r 0$. We call the elements of $B_e(\mathbf{s})$ Kleshchev multipartitions.
Finally, for $\lambda \in \mathcal{P}_r$ we define

$$\varphi_i(\lambda) := \max\left\{ j \geq 0 \mid \widetilde{f_i}^j(\lambda) \neq 0 \right\} \quad \text{and} \quad \varepsilon_i(\lambda) := \max\left\{ j \geq 0 \mid \widetilde{e_i}^j(\lambda) \neq 0 \right\}.$$

To complete the definition for $r = 0$ we define $B_e(\mathbf{0})$ to the be directed graph with exactly one vertex and no edges, where we denote by $\mathbf{0}$ the empty sequence and for consistency we set $\widetilde{\mathbb{Z}^0_{\geq 0}} := \{\mathbf{0}\}$.

Lemma 1.6 ([Ari02, Thm 11.8][DVV17, Sec 1.5, 2.3.2], [Kas93, Thm 3.3.1])**.** *Let $r \geq 0$ and $\mathbf{t} := (t_1, \ldots, t_r) \in \mathbb{Z}^r$. Then there is a well-defined $\widehat{\mathfrak{sl}}_e$-module $\mathcal{F}_{\mathbf{t}}$ called the* Fock space with multi-charge $\mathbf{t}$. *It is isomorphic to the Fock space $\mathcal{F}_{\mathbf{s}}$ for any $\mathbf{s} \in \widetilde{\mathbb{Z}_{\geq 0}}^r$ in $\widetilde{\mathbb{Z}_{\geq 0}}$ such that the multi-sets $\{t_1 \pmod e, \ldots, t_r \pmod e\}$ and $\{s_1 \pmod e, \ldots, s_r \pmod e\}$ are equal. The crystal graph associated to $\mathcal{F}_{\mathbf{s}}$ is isomorphic to $B_e(\mathcal{F}_{\mathbf{s}})$. As it is isomorphic to the crystal graph associated to $\mathcal{F}_{\mathbf{t}}$ we will also refer to this crystal graph as $B_e(\mathcal{F}_{\mathbf{t}})$.*

Proposition 1.7. *Let $\lambda \in P^+$ be a dominant integral weight and define $\overline{\lambda} \in \overline{P^+}$ as the unique element such that $\lambda \equiv \overline{\lambda} \pmod{\mathbb{Z}\partial}$. As the Λ_i are linearly independent, there exists a unique $r \geq 0$ and $\mathbf{s}_\lambda = (s_1, \ldots, s_r) \in \widetilde{\mathbb{Z}^r_{\geq 0}}$ such that $\overline{\lambda} = \Lambda_{s_1} + \cdots + \Lambda_{s_r}$. Then the crystal graph associated to the irreducible highest weight module $L(\lambda)$ is isomorphic to $B_e(\mathbf{s}_\lambda)$.*

Proof. Denote by $U_u(\widehat{\mathfrak{sl}}_e)$ the quantum enveloping algebra of $\widehat{\mathfrak{sl}}_n$, cf. [HK02]. Then there exists a quantinization $L_u(\lambda)$ of $L(\lambda)$ such that the specialisation of $L_u(\lambda)$ at $u = 1$ is isomorphic to $L(\lambda)$. To $L_u(\lambda)$, too, we can associate a directed graph via a *crystal basis* and this, too, is unique up to graph isomorphism, and also called a *crystal graph*.
There exists a crystal basis of $L_u(\lambda)$ such that its specialisation is a perfect basis of $L(\lambda)$, more precisely this is satisfied by an upper global basis of $L_u(\lambda)$, and it follows that the crystal graph associated to $L_u(\lambda)$ is isomorphic to that of $L(\lambda)$.
By [Ari02, Thm 11.11] the crystal graph of $L(\overline{\lambda})$ is exactly $B_e(\mathbf{s}_\lambda)$, so it remains to show that the crystal graphs of $L(\lambda)$ and $L(\overline{\lambda})$ are isomorphic.
If $r = 0$, this is trivial, as the irreducible highest weight module $L(k\partial)$ for an integer k has $\mathbb{C}$-dimension one and it is easy to see that the associated crystal graph is $B_e(\mathbf{0})$.
For $r \geq 1$ one can use the affinization in [HK02, Section 10.1] to show that the crystal graph of $L_u(\overline{\lambda})$ is isomorphic to that of $L_u(\overline{\lambda} + k\partial)$ for every integer k, and in particular the crystal graph of $L_u(\overline{\lambda})$ is isomorphic to that of $L_u(\lambda)$. The same clearly holds for their specialisations at $u = 1$. □

Corollary 1.8. *Let $M \in \mathcal{O}_{int}$ and suppose that $M \cong \oplus_j L(\lambda_j)$ for $\lambda_j \in P^+$ is a decomposition into irreducible highest weight modules. Then the crystal graph of M is isomorphic to $\coprod_j B_e(\mathbf{s}_{\lambda_j})$, where $\coprod$ denotes the disjoint union of graphs.*

Following this we can obtain information about modules in $\mathcal{O}_{\text{int}}$ via solely combinatorial observations, so we study the graphs $B_e(\mathbf{s})$ and $B_e(\mathcal{F}_{\mathbf{s}})$ in some more detail.

The first two results are well-known.

Lemma 1.9. *Let $r \geq 1$ and $\mathbf{s} \in \widetilde{\mathbb{Z}^r_{\geq 0}}$. If λ is a vertex in $B_e(\mathbf{s})$, then either $\lambda = \emptyset \vdash_r 0$ or λ has an i-good node.*

Lemma 1.10. *Let $r \geq 1$, $\lambda \vdash_r n$, and $0 \leq i \leq e-1$.*
The number of i-co-normal nodes of λ is exactly $\varphi_i(\lambda)$. Similarly, the number of i-normal nodes of λ is exactly $\varepsilon_i(\lambda)$.

The following is an easy but key observation on co-normal nodes:

Lemma 1.11. *Let $\lambda \vdash_r n$. Then λ has exactly r more addable than removable nodes. Hence, the number of co-normal nodes of λ is exactly r larger than the number of normal nodes.*
By Lemma 1.10 this is equivalent to saying that $\sum_i (\varphi_i(\lambda) - \varepsilon_i(\lambda)) = r$.

Proof. Consider the Young diagram of $\lambda^{(1)}$. If x is a removable node of $\lambda^{(1)}$ then there is an addable node in the row directly below x and we can pair of removable and addable nodes in this manner. But then we are left with the addable node in the very first row of $\lambda^{(1)}$, which does not have a removable node above it. Hence, $\lambda^{(1)}$ has exactly one more addable than removable node and the same holds true for $\lambda^{(2)}, \ldots, \lambda^{(r)}$, so in total λ has exactly r more addable than removable nodes.

Now for co-normal nodes:
At the beginning of the algorithm to determine all co-normal nodes all addable and removable nodes are considered at once, since every node is an i-node for some i. Then they are removed in pairs of addable and removable nodes, so the difference between their numbers remains constantly equal to r. □

We follow [Sha11, 5.1] for the definition of $\widehat{\mathfrak{sl}}_e$-categorification in the sense of [Rou08]:

Definition 1.12. *Set $q := \exp(2\pi\sqrt{-1}/e) \in \mathbb{C}$.*
Let C be a $\mathbb{C}$-linear artinian abelian category. For any functor $F : C \to C$ and any $X \in \mathrm{End}(F)$ we call the generalised eigenspace of X acting on F with eigenvalue $a \in \mathbb{C}$ the a-eigenspace of X in F.
Then an $\widehat{\mathfrak{sl}}_e$-categorification on C *consists of*

a) an adjoint pair (U, V) of exact functors $C \to C$,
b) $X \in \mathrm{End}(U)$ and $T \in \mathrm{End}(U^2)$, and
c) a decomposition $C = \oplus_{\lambda \in P} C_\lambda$,

satisfying the following: Set U_i (resp. V_i) to be the q^i-eigenspace of X in U (resp. in V) for $0 \leq i \leq e-1$. Then

i) it is $U = \oplus_{i=0}^{e-1} U_i$,
ii) the endomorphisms X and T satisfy the relations

$$(1_U T) \circ (T 1_U) \circ (1_U T) = (T 1_U) \circ (1_U T) \circ (T 1_U),$$
$$(T + 1_{U^2}) \circ T - q 1_{U^2} = 0,$$
$$T \circ (1_U X) \circ T = q X 1_U,$$

iii) the map $U_i \mapsto e_i$ and $V_i \mapsto f_i$ for $0 \leq i \leq e-1$ defines an integrable representation of $\widehat{\mathfrak{sl}}_e$ on the complexification $K_0(C) := \mathbb{C} \otimes_{\mathbb{Z}} R_0(C)$ of the Grothendieck group,
iv) $U_i(C_\lambda) \subseteq C_{\lambda+\alpha_i}$ and $V_i(C_\lambda) \subseteq C_{\lambda-\alpha_i}$, where α_i is the i'th simple root of $\widehat{\mathfrak{sl}}_e$,
v) V is isomorphic to a left adjoint of U.

We fix a $\mathbb{C}$-linear artinian abelian category C possessing an $\widehat{\mathfrak{sl}}_e$-categorification afforded by an adjoint pair of functors (U, V) and endomorphisms X and T.

Proposition 1.13 ([CR08, Prop 5.20], [Sha11, 6.2])**.** *Let $0 \leq i \leq e-1$. Then the data (U_i, V_i, X, T) yields an $\mathfrak{sl}_2$-categorification on C in the sense of* [CR08, Sec 5].

For an object $M \in \mathcal{C}$ set $\widetilde{U}_i(M) := \mathrm{soc}(U_i(M))$ and $\widetilde{V}_i(M) := \mathrm{head}(V_i(M))$. If S is simple, then $\widetilde{U}_i(S)$ is either simple or 0. Similarly, $\widetilde{V}_i(S)$ is either 0 or simple. Moreover, if $\widetilde{V}_i(S) \neq 0$, then the multiplicity of $\widetilde{V}_i(S)$ in $V_i(S)$ is exactly $\max_j\{j \geq 0 \mid \widetilde{V}_i^{\,j}(S) \neq 0\}$.

Proposition 1.14 ([Sha11, Prop 6.2])**.** *Let $\mathrm{Irr}(\mathcal{C})$ be the set of simple objects of $\mathcal{C}$ up to isomorphism. The triple*

$$(\{[S] \mid S \in \mathrm{Irr}(\mathcal{C})\},\ \{\widetilde{U}_i \mid 0 \leq i \leq e-1\},\ \{\widetilde{V}_i \mid 0 \leq i \leq e-1\})$$

is a perfect basis of $K_0(\mathcal{C})$.

These preliminaries allow us to prove our key result on $\widehat{\mathfrak{sl}}_e$-categorification:

Theorem 1.15. *Suppose the $\widehat{\mathfrak{sl}}_e$-module $K_0(\mathcal{C})$ is in $\mathcal{O}_{int}$ and the decomposition into irreducible highest weight modules is $\oplus_j L(\lambda_j)$ for $\lambda_j \in P^+$. Denote by $B(\mathcal{C})$ the crystal graph associated to its perfect basis from Proposition 1.14 and let*

$$\Psi : B(\mathcal{C}) \to \oplus_j B_e(\mathbf{s}_{\lambda_j})$$

a crystal graph isomorphism as in Corollary 1.8.
Let $S \in \mathcal{C}$ be simple and $\mathbf{s} \in \widetilde{\mathbb{Z}^r_{\geq 0}}$ such that $\Psi([S]) \in B_e(\mathbf{s})$.
Then $V(S)$ has at least r constituents.

Proof. Let $\ell_i(S) := \max_j\{j \geq 0 \mid \widetilde{V}_i^{\,j}(S) \neq 0\}$ for $0 \leq i \leq e-1$. Since $V = \oplus_i V_i$, we see that $V(S)$ has at least $\sum_i \ell_i(S)$ constituents by Proposition 1.13. As Ψ is a crystal isomorphism and $\Psi([S])$ is in $B_e(\mathbf{s})$, it follows from Lemma 1.11 that $\sum_i \ell_i(S) = \sum_i \varphi_i(\Psi([S]))$ is at least r. □

We also get an analogous result if the $\widehat{\mathfrak{sl}}_e$-module decomposes as a direct sum of Fock spaces via a nearly identical proof.

Corollary 1.16. *Suppose the $\widehat{\mathfrak{sl}}_e$-module $K_0(\mathcal{C})$ is in $\mathcal{O}_{int}$ and has a decomposition into Fock spaces $K_0(\mathcal{C}) \cong \bigoplus_{\mathbf{s}} \mathcal{F}_{\mathbf{s}}$ for $\mathbf{s} \in \widetilde{\mathbb{Z}^r_{\geq 0}}$ for $r \geq 1$. Denote by $B(\mathcal{C})$ the crystal graph associated to $K_0(\mathcal{C})$ via Proposition 1.14. Then there is a crystal graph isomorphism*

$$\Psi : B(\mathcal{C}) \to \oplus_{\mathbf{s}} B_e(\mathcal{F}_{\mathbf{s}}).$$

If $S \in \mathcal{C}$ is simple and $\Psi([S])$ is in $B_e(\mathcal{F}_{\mathbf{s}})$ for some $\mathbf{s} = (s_1, \ldots, s_r)$, then $V(S)$ has at least r constituents.

1.2. **Rational cyclotomic Cherednik algebras.** Fix integers $r \geq 1$ and $e \geq 2$ and let $\mathbf{t} = (t_1, \ldots, t_r) \in \mathbb{Z}^r$. Then for every non-negative integer n we can define a *cyclotomic rational Cherednik algebra (or cyclotomic rational double affine Hecke algebra)* $\mathcal{H}_{n,\mathbf{t},e}$: It is a quotient of the smash product of the complex group algebra $\mathbb{C}[W]$, where W is a complex reflection group of type $G(r, 1, n)$, with the tensor algebra of $N \oplus N^*$, where N is the n-dimensional vector space on which W acts naturally, cf. [Sha11, 3.1] for a precise definition.
For every such rational Cherednik algebra there exists a module category $\mathcal{O}_{n,\mathbf{t},e}$ consisting of all $\mathcal{H}_{n,\mathbf{t},e}$-modules that are finitely generated and acted locally nilpotently on by a certain subalgebra of $\mathcal{H}_{n,\mathbf{t},e}$.
In the following we set $\mathcal{H}_n := \mathcal{H}_{n,\mathbf{t},e}$ and $\mathcal{O}_n := \mathcal{O}_{n,\mathbf{t},e}$.
Bezrukavnikov and Etingof defined parabolic induction and restriction functors $\mathcal{O}_n \to \mathcal{O}_{n+1}$ and $\mathcal{O}_n \to \mathcal{O}_{n-1}$ (cf. [BE09]) which we will denote by Ind_n and Res_n, respectively, in the following. Set $\mathcal{O} := \oplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}_n$ and

$$\mathrm{Res} := \oplus_{n \geq 0} \mathrm{Res}_n \quad \text{and } \mathrm{Ind} := \oplus_{n \geq 0} \mathrm{Ind}_n .$$

Proposition 1.17 ([Sha11, Cor 4.5])**.** *Category $\mathcal{O}$ possesses an $\widehat{\mathfrak{sl}}_e$-categorification for which the pair of adjoint functors is given by* (Res, Ind) *and the $\widehat{\mathfrak{sl}}_e$-module $K_0(\mathcal{O})$ is in category $\mathcal{O}_{int}$. As an $\widehat{\mathfrak{sl}}_e$-module, $K_0(\mathcal{O})$ is isomorphic to the Fock spacce $\mathcal{F}_{\mathbf{t}}$.*

By Corollary 1.16 this implies the following:

Theorem 1.18. *Let $0 \neq M \in \mathcal{O}$. Then* $\mathrm{Ind}(M)$ *has at least r constituents. In particular, if $r \geq 2$, then* $\mathrm{Ind}(M)$ *is always reducible.*

Remark 1.19. *We can strengthen the result at least on a sub-class of modules by using what we will later prove in Proposition 2.11: Let $\mathbf{H}_n$ be the Ariki-Koike algebra associated to $G(r, 1, n)$ with parameters $q := \exp(2\pi\sqrt{-1}/e)$ and $Q_i := q^{t_i}$ (cf. Chapter 2 for a precise definition). Then there exists a well-known exact functor $\mathrm{KZ}_n : \mathcal{O}_n \to \mathbf{H}_n$-mod and the number of constituents of $\mathrm{KZ}_n(M)$ is a lower bound for the number of constituents of M for any $M \in \mathcal{O}$. Hence, the combination of* [Sha11, Cor 2.3, Le 2.6] *with Proposition 2.11 yields the following: If $n \geq 1$ and $M \in \mathcal{O}_n$ such that $\mathrm{KZ}_n(M) \neq 0$, then $\mathrm{Ind}_n(M)$ has at least $r + 1$ constituents.*

2. Parabolic induction on Ariki-Koike algebras

In the following we will use our previous results on crystal graphs to give a new lower bound on the number of constituents of parabolically induced modules of Ariki-Koike algebras. After some preliminaries on the object at hand we first reduce the problem to the case of so-called q-connected parameters. Once this is done, we have to differentiate the cases that the parameter q is either 1 or not equal to 1. In the second case we can mostly apply our earlier results on $\widehat{\mathfrak{sl}}_e$-categorification, whereas for $q = 1$ some manual computation yields the corresponding result. Note that we treat the case of a generic parameter independently, as we do not have an $\widehat{\mathfrak{sl}}_e$-categorification result in this case, but we are still able to use our results on the crystal graphs $B_e(\mathbf{s})$.
We close with the analogous result for the closely related degenerate cyclotomic Hecke algebra.

2.1. **Preliminaries.** Throughout this chapter let K be a field. For convenience we assume K to be algebraically closed.

We begin with some preliminaries:
Let n and r be positive integers and $q, Q_1, \ldots, Q_r$ invertible elements of K. Then the *Ariki-Koike algebra* $\mathbf{H}_{n,r}(q; Q_1, \ldots, Q_r)$ is the unital associative K-algebra with generators $T_0, \ldots, T_{n-1}$ satisfying relations

$$\begin{aligned} (T_0 - Q_1)\cdots(T_0 - Q_r) &= 0 \\ (T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leq i \leq n-1 \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && \text{for } 1 \leq i \leq n-2 \\ T_iT_j &= T_jT_i && \text{for } |i-j| > 1. \end{aligned}$$

This algebras has been defined by Ariki-Koike and, independently, by Broué-Malle, cf. [AK94, BM93]. It is obvious that a re-ordering of the Q_i does not change the resulting algebra. In the following we fix parameters and set $\mathbf{H}_n := \mathbf{H}_{n,r}(q; Q_1, \ldots, Q_r)$.
It is well-known that the elements $T_1, \ldots T_{n-1}$ generate an Iwahori-Hecke algebra of type A_{n-1} with parameter q. Hence, as usual, for $w \in \mathfrak{S}_n$ we set $T_w := T_{i_1} \cdots T_{i_k}$ whenever $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression of w in the generators $s_i := (i, i+1)$. We define

the *Jucys-Murphy elements* L_i inductively by setting $L_1 := T_0$ and $L_{i+1} := q^{-1}T_iL_iT_i$ for $1 \leq i \leq n-1$. It has been shown by Ariki and Koike in [AK94] that

$$\left\{L_1^{a_1} \cdots L_n^{a_n} T_w \mid w \in \mathfrak{S}_n, 0 \leq a_1, \ldots, a_n \leq r-1\right\}$$

is a K-basis of $\mathbf{H}_n$.

This implies that $\mathbf{H}_n$ is a subalgebra of $\mathbf{H}_{n+1}$ and that $\mathbf{H}_{n+1}$ is free as a left $\mathbf{H}_n$-module. Hence, there exists an exact induction functor

$$\mathrm{Ind}_n := \mathrm{Ind}_{\mathbf{H}_n}^{\mathbf{H}_{n+1}} : \mathbf{H}_n\text{-mod} \to \mathbf{H}_{n+1}\text{-mod}; M \mapsto M \otimes_{\mathbf{H}_n} \mathbf{H}_{n+1} .$$

As this functor is exact it yields a homomorphism of Grothendieck groups

$$R_0(\mathbf{H}_n) \to R_0(\mathbf{H}_{n+1}); [M] \mapsto [\mathrm{Ind}_n(M)],$$

where $[M]$ is the class of the $\mathbf{H}_n$-module M in the Grothendieck group. By slight abuse of notation we denote this homomorphism, too, by Ind_n.

For every r-multipartition $\lambda \vdash_r n$ there exists a well-defined finite dimensional $\mathbf{H}_n$-module S^λ called a *Specht module*, defined in [DJM98].
On each Specht module S^λ there exists a well-defined bilinear form whose radical $\mathrm{rad}\, S^\lambda$ is an $\mathbf{H}_n$-submodule of S^λ and we set $D^\lambda := S^\lambda / \left(\mathrm{rad}\, S^\lambda\right)$. These modules fit neatly into the concept of viewing $\mathbf{H}_n$ as a cellular algebra and in [DJM98] it is shown that the set

$$\left\{D^\lambda \mid \lambda \vdash_r n,\ D^\lambda \neq 0\right\}$$

is a complete set of pairwise non-isomorphic $\mathbf{H}_n$-modules.
Furthermore, they deduce that the Grothendieck group $R_0(\mathbf{H}_n)$ is generated by $\left\{[S^\lambda] \mid \lambda \vdash_r n\right\}$.

2.2. **Reduction to q-connected parameter sets.** Many questions on the representation theory of Ariki-Koike algebras have only been covered for so-called q-connected parameter sets.

Definition 2.1. *Two elements x and y of K are called* q-connected *if there exists an integer k such that $x = q^k y$. We write $x \sim_q y$. Clearly, this defines an equivalence relation on K. We call a set or sequence X with elements in K q-connected if all elements of X are q-connected. Finally, if X and Y are q-connected sets (or sequences) over K we say that X and Y are q-connected if there exist elements $x \in X$ and $y \in Y$ such that x and y are q-connected.*

We set $\mathbf{Q} := (Q_1, \ldots, Q_r)$. As reordering of the Q_i does not change the algebra $\mathbf{H}_n$ we can assume without loss of generality that

$$\mathbf{Q} = \mathbf{Q}_1 \coprod \cdots \coprod \mathbf{Q}_t,$$

for q-connected sequences $\mathbf{Q}_i$ which are pairwise not q-connected, where $\coprod$ denotes the concatenation of sequences. In particular, t is the number of $\sim_q$-equivalence classes on $\mathbf{Q}$. For $1 \leq j \leq t$ define $r_j := |\mathbf{Q}_j|$, the length of $\mathbf{Q}_j$. Throughout this section we denote by $\otimes$ the tensor product over K.

Theorem 2.2 ([DM02, Thm 1.1])**.** *There is a Morita equivalence*

$$\mathbf{H}_n \sim_{Morita} \mathbf{H}_n^t := \bigoplus_{\substack{0 \leq n_1, \ldots, n_t \leq n, \\ \sum_i n_i = n}} {}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t},$$

where ${}^j\mathbf{H}_m := \mathbf{H}_{m,\,r_j}(q, \boldsymbol{Q}_j)$ *for* $0 \le m \le n$ *and* $1 \le j \le t$*. In particular, there is an exact functor*

$$F_n : \mathbf{H}_n\text{-mod} \to \mathbf{H}_n^t\text{-mod}.$$

Remark 2.3. *As the functor* F_n *is exact it induces a homomorphism on the corresponding Grothendieck groups. By abuse of notation we will denote it, too, by* F_n*.*
Note that the Grothendieck group of $\mathbf{H}_n^t$ *is the direct sum of the Grothendieck groups of the algebras* ${}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}$*. The irreducible modules of* ${}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}$ *are exactly the tensor products of irreducible modules of* ${}^1\mathbf{H}_{n_1}, \ldots, {}^t\mathbf{H}_{n_t}$*.*

To study the functor F_n we first consider its images on Specht modules.

Proposition 2.4 ([DM02, Prop 4.11])**.** *Let* $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ *be an r-multipartition of n. Then define* ${}^1\lambda$ *to be the* r_1*-multipartition consisting of the first* r_1 *components of* λ*, i.e.* ${}^1\lambda := (\lambda^{(1)}, \ldots, \lambda^{(r_1)})$*. Then let* ${}^2\lambda$ *be the* r_2*-multipartition consisting of the next* r_2 *components of* λ*, i.e.* ${}^2\lambda = (\lambda^{(r_1+1)}, \ldots, \lambda^{(r_1+r_2)})$*,etc. In the end we have* $\lambda = {}^1\lambda \coprod \cdots \coprod {}^t\lambda$*. Then for the Specht module* S^λ *it is*

$$F_n(S^\lambda) \cong S^{{}^1\lambda} \otimes \cdots \otimes S^{{}^t\lambda},$$

which is an $\mathbf{H}_n^s$*-module on which nearly all direct summands act as zero with the exception of* ${}^1\mathbf{H}_{|{}^1\lambda|} \otimes \cdots \otimes {}^t\mathbf{H}_{|{}^t\lambda|}$ *.*
A completely analogous result holds for the module D^λ*, where we just replace every* S *by* D*.*

The induction Ind_n on Specht modules is well-understood by the following result by Mathas:

Proposition 2.5 ([Mat09, Thm A])**.** *Let* λ *be an r-multipartition of n. The induced module* $\mathrm{Ind}_n(S^\lambda)$ *has a filtration* $0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_a = \mathrm{Ind}_n(S^\lambda)$ *such that for all* $1 \le j \le a$ *the quotient* I_j/I_{j-1} *is also a Specht-module. Moreover, the Specht modules appearing as such quotients* I_i/I_{i-1} *are exactly those indexed by the multipartitions of* $n+1$ *obtained by adding exactly one addable node to* $[\lambda]$ *and every such multipartition appears exactly once.*

We now move towards an analogous result for $\mathbf{H}_n^t$, at least on the level of Grothendieck groups. This requires the definition of a number of homomorphisms $R_0(\mathbf{H}_n^t) \to R_0(\mathbf{H}_{n+1}^t)$: For $0 \le n_1, \ldots, n_t \le n$ with $\sum_i n_i = n$ and $1 \le j \le t$ we define

$${}^j\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)} : R_0\left(\mathbf{H}_n^t\right) \to R_0\left(\mathbf{H}_{n+1}^t\right)$$

by giving its image on classes of irreducible modules. If M is an irreducible module of $\mathbf{H}_n^t$ that is not in $\left({}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}\right)$-mod, then set ${}^j\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}([M]) := 0$. If M is an irreducible module of $\mathbf{H}_n^t$ and in $\left({}^1\mathbf{H}_{n_1} \otimes \cdots \otimes {}^t\mathbf{H}_{n_t}\right)$-mod, then it is isomorphic to the tensor product $D_1 \otimes \cdots \otimes D_t$ for irreducible ${}^i\mathbf{H}_{n_i}$-modules D_i. In this case we set

$${}^j\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}([M]) := [D_1 \otimes \cdots \otimes D_{j-1} \otimes \left(\mathrm{Ind}_{{}^j\mathbf{H}_{n_j}}^{{}^j\mathbf{H}_{n_j+1}}\left(D_j\right)\right) \otimes D_{j+1} \otimes \cdots \otimes D_t],$$

i.e. we apply the usual parabolic induction in the j-th component.
By Proposition 2.5 this immediately yields the following:

Lemma 2.6. *For* $1 \le i \le t$ *with* $i \ne j$ *let* $M_i \in {}^i\mathbf{H}_{n_i}$-mod*. Let* $\alpha \vdash_{r_j} n_j$ *be a multipartition. Then it is*

$${}^j\mathrm{Ind}_{n,t}^{(n_1,\ldots,n_t)}([M_1 \otimes \cdots \otimes S^\alpha \otimes \cdots \otimes M_t]) = \sum_{\substack{\beta \vdash_{r_j} n_j+1,\\ |\beta \setminus \alpha| = 1}} \left[M_1 \otimes \cdots \otimes S^\beta \otimes \cdots \otimes M_t\right],$$

i.e. all partitions β obtained by adding exactly one node to α appear exactly once.

Now we set

$$\mathrm{Ind}_{n,t}^{(n_1,\dots,n_t)} := \sum_{j=1}^{t} {}^{j}\,\mathrm{Ind}_{n,t}^{(n_1,\dots,n_t)}$$

and finally

$$\mathrm{Ind}_{n,t} := \sum_{\substack{0\le n_1,\dots,n_t\le n,\\ \sum_i n_i=n}} \mathrm{Ind}_{n,t}^{(n_1,\dots,n_t)}.$$

Theorem 2.7. *The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbf{H}_n) & \xrightarrow{F_n} & R_0(\mathbf{H}_n^t) \\ \downarrow{\scriptstyle \mathrm{Ind}_n} & & \downarrow{\scriptstyle \mathrm{Ind}_{n,t}} \\ R_0(\mathbf{H}_{n+1}) & \xrightarrow{F_{n+1}} & R_0\left(\mathbf{H}_{n+1}^t\right) \end{array}$$

Proof. Let $\lambda \vdash_r n$ and define ${}^1\lambda,\dots,{}^t\lambda$ as in Proposition 2.4. For $1\le i\le t$ set $n_i := |{}^i\lambda|$. By definition it is $\mathrm{Ind}_{n,t}([F_n(S^\lambda)]) = \mathrm{Ind}_{n,t}^{(n_1,\dots,n_t)}([F_n(S^\lambda)])$ and by Lemma 2.6 and the definition of $\mathrm{Ind}_{n,t}^{(n_1,\dots,n_t)}$ we have

$$\mathrm{Ind}_{n,t}^{(n_1,\dots,n_t)}([F_n(S^\lambda)]) = \sum_{j=1}^{t} \sum_{\substack{\beta\vdash_{r_j} n_j+1\\ |\beta\setminus{}^j\lambda|=1}} [S^{{}^1\lambda}\otimes\cdots\otimes S^{{}^{j-1}\lambda}\otimes S^\beta\otimes S^{{}^{j+1}\lambda}\otimes\cdots\otimes S^{{}^t\lambda}].$$

For $1\le j\le t$ and $\beta\vdash_{r_j} n_j$ with $|\beta\setminus{}^j\lambda| = 1$ let $\mu(j,\beta)\vdash_r n$ be the multipartition of $n+1$ obtained as the concatenation $({}^1\lambda,\dots,\beta,\dots,{}^t\lambda)$, where β is the j'th subsequence. By Proposition 2.4 it is $F_{n+1}([S^{\mu(j,\beta)}]) = [S^{{}^1\lambda}\otimes\cdots\otimes S^{{}^{j-1}\lambda}\otimes S^\beta\otimes S^{{}^{j+1}\lambda}\otimes\cdots\otimes S^{{}^t\lambda}]$.
Clearly, the $\mu(j,\beta)$ run exactly over all multipartitions of $n+1$ which are obtained from λ by adding exactly one node and every such multipartition appears exactly once. Hence, by Proposition 2.5 it is

$$\mathrm{Ind}_n([S^\lambda]) = \sum_{j=1}^{t}\sum_{\beta\vdash_{r_j} n_j+1} [S^{\mu(j,\beta)}]$$

and thus $F_{n+1}(\mathrm{Ind}_n([S^\lambda])) = \mathrm{Ind}_n^t(F_n([S^\lambda]))$.
Since the classes of Specht modules generate the Grothendieck group of $\mathbf{H}_n$ this already implies the commutativity of the diagram. □

Corollary 2.8. *The homomorphisms F_n and F_{n+1} obtained from the Morita equivalence preserve the number of irreducible constituents. Hence, if $\lambda = (\lambda^{(1)},\dots,\lambda^{(r)})$ is an r-multipartition of n such that $D^\lambda \neq 0$, then the number of irreducible constituents of the module $\mathrm{Ind}_n(D^\lambda)$ is equal to that of $\mathrm{Ind}_{n,t}\left(D^{{}^1\lambda}\otimes\cdots\otimes D^{{}^t\lambda}\right)$ by Proposition 2.4. By definition, this is equal to the number obtained by summing the number of constituents of $\mathrm{Ind}_{{}^j\mathbf{H}_{n_j}}^{{}^j\mathbf{H}_{n_j+1}}(D^{{}^j\lambda})$ over all j, where $n_j := |{}^j\lambda|$.*
Since ${}^j\mathbf{H}_{n_j}$ and ${}^j\mathbf{H}_{n_j+1}$ are defined over q-connected parameters $\boldsymbol{Q}_j$ we have now reduced the problem of finding the number of constituents of induced modules to the q-connected parameter case.

Remark 2.9. *A completely analogous result holds for restriction in place of induction. The filtration in Proposition 2.5 has to be replaced by that in* [Mat18] *for restriction of Specht modules. One has to pay attention when defining the partial restriction homomorphisms* ${}^{j}\operatorname{Res}_{n,t}^{(n_1,\dots,n_t)}$*; they are only defined for* $n_j \geq 1$*. Then they are defined on classes of irreducibles via the usual parabolic restriction in the* j*'th component of the tensor product. In total, we define the restriction on the* $\mathbf{H}_n^t$ *side to be*

$$\operatorname{Res}_{n,t} = \bigoplus_{j=1}^{t} \bigoplus_{\substack{0 \leq n_1,\dots,n_t \leq n, \\ \sum_i n_i = n \\ n_j \geq 1}} {}^{j}\operatorname{Res}_{n,t}^{(n_1,\dots,n_t)}.$$

Then the following diagram commutes:

$$\begin{array}{ccc} R_0\left(\mathbf{H}_n\right) & \xrightarrow{F_n} & R_0\left(\mathbf{H}_n^t\right) \\ \Big\downarrow{\scriptstyle \operatorname{Res}_n} & & \Big\downarrow{\scriptstyle \operatorname{Res}_{n,t}} \\ R_0\left(\mathbf{H}_{n-1}\right) & \xrightarrow{F_{n-1}} & R_0\left(\mathbf{H}_{n-1}^t\right) \end{array}$$

2.3. **q-connected parameters.** Assume that $\mathbf{Q} = (Q_1, \dots, Q_r)$ is q-connected, as we have just reduced our problem to this case.
We can further simplify the setting without loss of generality: Let $a \in K^*$. Then $\mathbf{H}_{n,r}(q; Q_1, \dots, Q_r)$ is isomorphic to $\mathbf{H}_{n,r}(q; aQ_1, \dots, aQ_r)$ by replacing T_0 with $a^{-1}T_0$. Hence, if $\mathbf{Q}$ is q-connected we can assume without loss of generality that there exist non-negative integers $s_1, \dots, s_r$ such that $Q_i = q^{s_i}$ for $1 \leq i \leq r$, and will assume this to be the case from now on. Now let $e \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be the multiplicative order of q in K^*. Then we can additionally assume that $0 \leq s_1, \dots, s_r < e$ and as reordering of the Q_i does not change $\mathbf{H}_n$ we also assume $s_1 \leq s_2 \leq \dots \leq s_r$. Set $\mathbf{s} := (s_1, \dots, s_r) \in \widetilde{\mathbb{Z}_{\geq 0}}$.

In the following we will have to differentiate the cases $q \neq 1$ and $q = 1$.
Note that in the latter case people will often switch to considering so-called degenerate cyclotomic Hecke algebras instead, which are slightly different than the Ariki-Koike algebras for $q = 1$ we consider here, but the definitions of Ariki-Koike algebras make sense, too, for $q = 1$, so we see no reason to exclude this case. However, for completeness we will remark in Theorem 2.24 how to handle degenerate cyclotomic Hecke algebras.

2.3.1. *The case* $q \neq 1$. Set $\mathbf{H}\text{-mod} := \bigoplus_{n \geq 0} \mathbf{H}_n\text{-mod}$ and

$$\operatorname{Ind} := \bigoplus_{n \geq 0} \operatorname{Ind}_n \quad \operatorname{Res} := \bigoplus_{n \geq 0} \operatorname{Res}_n .$$

We first consider the case $e < \infty$. This is where our previous results do come in:

Proposition 2.10 ([Sha10, Exp 5.2.5], [Ari02, Thm 12.5], [Ari06, Thm 6.1])**.** *The functors* Res *and* Ind *constitute a pair of bi-adjoint functors on* $\mathbf{H}$-mod*, yielding an* $\widehat{\mathfrak{sl}}_e$*-categorification. The* $\widehat{\mathfrak{sl}}_e$*-module* $K_0(\mathbf{H}\text{-mod})$ *is isomorphic to the irreducible highest weight module* $L(\Lambda)$ *with* $\Lambda = \Lambda_{s_1} + \dots + \Lambda_{s_r}$*. If we denote by* $B_e(\mathbf{H})$ *the crystal graph associated to* $K_0(\mathbf{H}\text{-mod})$*, then* $B_e(\mathbf{H})$ *is isomorphic to* $B_e(\mathbf{s})$ *and the pre-image of the vertex* $\emptyset \vdash_r 0$ *is the class of the trivial module of the trivial algebra* $\mathbf{H}_0$*.*

Proposition 2.11. *Suppose* $2 \leq e < \infty$*. Let* $n \geq 1$ *and* $0 \neq M \in \mathbf{H}_n$-mod*. Then* $\operatorname{Ind}_n(M)$ *has at least* $r + 1$ *irreducible constituents. In particular,* $\operatorname{Ind}_n(M)$ *is reducible.*

Proof. Since Ind is exact it suffices to consider the case that M is irreducible. Let $\Psi : B_e(\mathbf{H}) \to B_e(\mathbf{s})$ be the crystal graph isomorphism from Proposition 2.10. By Proposition

1.13 and the fact that Ψ is a crystal isomorphism we see that the number of constituents of $\mathrm{Ind}_n(M)$ is at least $\sum_{i=0}^{e-1} \varphi_i(\Psi([M]))$. Since $\Psi([M])$ is not the empty partition, by Lemmas 1.9, 1.10, and 1.11 this number is at least $r+1$. □

Now suppose $e = \infty$.
This case has been studied in detail by Vazirani in [Vaz02]. Here, too, we obtain a result using crystal graphs. The crystal graph $B_\infty(\mathbf{s})$ is defined just as for $e < \infty$, if we define $x \equiv y \pmod{\infty}$ if and only if $x = y$ for integers x and y to extend the definition of the residue to the case $e = \infty$. Note that B_∞ is a crystal graph for $U_u(\widehat{\mathfrak{sl}}_\infty)$.
For $i \geq 0$ there exists a refined functor $i\text{-}\mathrm{Ind}_n\ \mathbf{H}_n\text{-mod} \to \mathbf{H}_{n+1}\text{-mod}$, defined via taking generalised eigenspaces of Jucys-Murphy elements, cf. e.g. [Ari06] for a definition. We define $i\text{-}\mathrm{Ind} := \oplus_{n\geq 0}\, i\text{-}\mathrm{Ind}_n$.

Proposition 2.12 ([Gro99, Vaz02])**.** *The functors i-Ind satisfy the following:*

a) i-Ind *is exact.*
b) *For* $M \in \mathbf{H}_n$-mod *it is* $\mathrm{Ind}(M) \cong \oplus_{i\geq 0}\, i\text{-}\mathrm{Ind}(M)$.
c) *Let* $M \in \mathrm{Irr}(\mathbf{H})$. *Then* $\widetilde{f}_i(M) := head(i\text{-}\mathrm{Ind}(M))$ *is either* 0 *or irreducible.*
d) *If* $M \in \mathrm{Irr}(\mathbf{H})$ *and* $\widetilde{f}_i(M) \neq 0$, *then the multiplicity of* $\widetilde{f}_i(M)$ *in* i-$\mathrm{Ind}(M)$ *is exactly* $\max_j\{j \geq 0 \mid \widetilde{f}_i(M)^j \neq 0\}$.
e) *Define a directed graph* $B_\infty(\mathbf{H})$ *with vertex set* $\mathrm{Irr}(\mathbf{H})$ *and directed edges* $M \xrightarrow{i} N$ *for* $M \in \mathrm{Irr}(\mathbf{H}_n)$ *and* $N \in \mathrm{Irr}(\mathbf{H}_{n+1})$ *if and only if* $\widetilde{f}_i(M) = N$. *Then* $B_\infty(\mathbf{H})$ *is isomorphic to* $B_\infty(\mathbf{s})$ *and the pre-image of the vertex* $\emptyset \vdash_r 0$ *is the class of the trivial module of the trivial algebra* $\mathbf{H}_0$.

With this in mind the proof of the following is completely analogous to the case $e < \infty$, as Lemmas 1.9, 1.10, and 1.11 all also hold for $e = \infty$.

Proposition 2.13. *Suppose* $e = \infty$. *Let* $n \geq 1$ *and* $0 \neq M \in \mathbf{H}_n$-mod. *Then* $\mathrm{Ind}_n(M)$ *has at least* $r+1$ *irreducible constituents. In particular,* $\mathrm{Ind}_n(M)$ *is reducible.*

2.3.2. *The case* $q = 1$. Now assume $q = 1$ and $n \geq 1$.
Note that the q-connectedness of the Q_i then implies that they are all equal to 1. Beware that in general $\mathbf{H}_n$ is *not* isomorphic to the so-called degenerate cyclotomic Hecke algebra.

For $q = 1$, the subalgebra of $\mathbf{H}_n$ that is generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the group algebra $K[\mathfrak{S}_n]$ and we identify the two. The Specht and irreducible modules of $\mathbf{H}_n$ for $q = 1$ have been studied by Mathas. Their structure is not overly complicated:

Proposition 2.14 ([Mat98, Theorem 3.7, Lemma 3.3])**.** *Let* $\lambda = \left(\lambda^{(1)}, \ldots, \lambda^{(r)}\right) \vdash_r n$ *be an r-multipartition such that* $D^\lambda \neq 0$. *Then the following holds:*

a) *It is* $\lambda^{(j)} = \emptyset$ *unless* $j = r$.
b) *The Jucys-Murphy elements* $L_1, \ldots, L_n$ *act trivially on* S^λ *and hence also on* D^λ.

Hence, the action on Specht and irreducible modules is completely determined by the restriction to the group algebra $K[\mathfrak{S}_n]$. In particular, the irreducible $\mathbf{H}_n$-modules are exactly the irreducible $K[\mathfrak{S}_n]$-modules seen as $\mathbf{H}_n$-modules by letting $T_0 = L_1$ act as 1.
For a partition $\alpha \vdash n$ denote by S^α the Specht module of $K[\mathfrak{S}_n]$ and as usual by D^α its quotient by the radical of the corresponding bilinear form. It is well-known that $D^\alpha \neq 0$ if and only if α is p-restricted, where p is the characteristic of K.

Denote by $\mathrm{Res}_{\mathfrak{S}_n}$ the restriction functor $\mathrm{Res}^{\mathbf{H}_n}_{K[\mathfrak{S}_n]}$. Then the next result follows from a close study of the explicit construction of Specht modules for $\mathbf{H}_n$ and $K[\mathfrak{S}_n]$.

Lemma 2.15. *Let* $\lambda = (\emptyset, \ldots, \emptyset, \lambda^{(r)}) \vdash_r n$. *Then the following holds:*

a) The restriction $\operatorname{Res}_{\mathfrak{S}_n}(S^\lambda)$ *is isomorphic to* $S^{\lambda^{(r)}} \in K[\mathfrak{S}_n]$-mod.
b) The restriction $\operatorname{Res}_{\mathfrak{S}_n}(D^\lambda)$ *is isomorphic to* $D^{\lambda^{(r)}} \in K[\mathfrak{S}_n]$-mod.

Proof. We refer the reader to [Mat04, Ch 3] and [Mat99, Ch 3] for details on the construction of Specht modules and follow these references. In particular, we do not vigorously define everything in this proof but rather assume familiarity with the construction and the necessary vocabulary.
Let us recall, though, that a tableau of shape $\mu \vdash_r n$ is a one-to-one labeling of the nodes of μ with the numbers $\{1, \dots, n\}$ and that a standard tableau is a special tableau. Accordingly, one can define (standard) tableaux of shape $\beta \vdash n$.
Set $\alpha := \lambda^{(r)}$. To shorten notation throughout this proof let $\mathbf{H} := \mathbf{H}_n$ and $\mathbf{h} := K[\mathfrak{S}_n]$.
Then $\mathbf{H}$ has a K-basis $M := \{m_{\mathfrak{u}\mathfrak{v}} \mid \mu \vdash_r n,\ \mathfrak{u}, \mathfrak{v} \text{ standard tableaux of shape } \mu\}$ and $\mathbf{h}$ has a K-basis $M' := \{m'_{\mathfrak{a}\mathfrak{b}} \mid \beta \vdash n,\ \mathfrak{a}, \mathfrak{b} \text{ standard tableaux of shape } \beta\}$. The key observation is that via identifying $\beta \vdash n$ with $(\emptyset, \dots, \emptyset, \beta) \vdash_r n$ the elements $m'_{\mathfrak{a}\mathfrak{b}}$ and $m_{\mathfrak{a}\mathfrak{b}}$ are equal for all standard tableaux $\mathfrak{a}$ and $\mathfrak{b}$ of shape β, hence M' embeds into M.
Now let $\mathbf{H}^\lambda$ be the K-span of all $m_{\mathfrak{u}\mathfrak{v}}$ where the shape of $\mathfrak{u}$ strictly dominates λ and let $\mathbf{h}^\alpha$ be the K-span of all $m'_{\mathfrak{a}\mathfrak{b}}$ where the shape of $\mathfrak{a}$ strictly dominates α. Note that in the first case we consider the dominance order of r-multipartitions, whereas in the latter the dominance order of ordinary partitions is used. Then $\mathbf{H}^\lambda \trianglelefteq \mathbf{H}$ and $\mathbf{h}^\alpha \trianglelefteq \mathbf{h}$ are two-sided ideals. Furthermore, it follows from the definitions that $\mathbf{H}^\lambda \cap \mathbf{h} = \mathbf{h}^\alpha$.
Now S^λ is a submodule of $\mathbf{H} / \mathbf{H}^\lambda$ with K-basis $\{m_{\mathfrak{u}} + \mathbf{H}^\lambda \mid \mathfrak{u} \text{ a standard tableau of shape } \lambda\}$, where we set $m_{\mathfrak{u}} := m_{\mathfrak{t}^\lambda \mathfrak{u}}$ for $\mathfrak{t}^\lambda$ the tableau obtained by labeling λ left to right, top to bottom. Similarly, via the above identification of tableau of shape α with those of shape λ we know that S^α is the submodule of $\mathbf{h} / \mathbf{h}^\alpha$ with K-basis $\{m_{\mathfrak{u}} + \mathbf{h}^\alpha \mid \mathfrak{u} \text{ a standard tableau of shape } \lambda\}$. Since the representatives of the basis elements all lie in $\mathbf{h}$ and because $\mathbf{H}^\lambda \cap \mathbf{h} = \mathbf{h}^\alpha$, we see that the K-vector space isomorphism $\Psi : S^\lambda \to S^\alpha ;\ m_{\mathfrak{u}} + \mathbf{H}^\lambda \mapsto m_{\mathfrak{u}} + \mathbf{h}^\alpha$ is an $\mathbf{h}$-module isomorphism. This proves a).
The irreducible modules D^λ and D^α are defined as quotients of S^λ and S^α by the radical of bilinear forms $\langle\ ,\ \rangle_\lambda$ and $\langle\ ,\ \rangle_\alpha$ on S^λ and S^α, respectively. These forms are defined via a number of of equations in $\mathbf{H} / \mathbf{H}^\lambda$ and $\mathbf{h} / \mathbf{h}^\alpha$, respectively, and from $\mathbf{H}^\lambda \cap \mathbf{h} = \mathbf{h}^\alpha$ we can deduce that Ψ respects the forms, i.e. $\langle x, y\rangle_\lambda = \langle \Psi(x), \Psi(y)\rangle_\alpha$ for all x, y in S^λ. Thus, Ψ induces an $\mathbf{h}$-isomorphism $D^\lambda \to D^\alpha$, proving b). □

Remark 2.16. *The above proof does not require q to be* 1. *As everything in* [Mat99] *is actually carried out for arbitrary Iwahori-Hecke algebras of type A, our proof still holds for arbitrary q, in which case the subalgebra of $\mathbf{H}_n$ generated by $T_1, \dots, T_{n-1}$ is an Iwahori-Hecke algebra of type A with parameter q.*
Finally, note that the Specht modules defined in [Mat99] *are what other authors might call* dual Specht modules *instead.*

As the symmetric group $\mathfrak{S}_n$ acts on the Jucys-Murphy elements by permuting the indices and the latter act trivially on simple $\mathbf{H}_n$-modules, one can prove the following:

Proposition 2.17. *Let D^λ be an irreducible $\mathbf{H}_n$-module for a multipartition $\lambda = (\emptyset, \dots, \emptyset, \lambda^{(r)}) \vdash_r n$ and B a K-basis of D^λ. Furthermore, denote by Y the set of distinguished right coset representatives of $\mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$, i.e. the set of right coset representatives with minimal length.*
Then

$$\left\{ b \otimes_{\mathbf{H}_n} L_{n+1}^j y \mid b \in B,\ y \in Y,\ 0 \le j < r \right\}$$

is a K-basis of $\mathrm{Ind}_n(D^\lambda)$.
For $0 \leq \ell < r$ *let* M_ℓ *be the K-vector space spanned by*

$$\left\{ b \otimes_{\mathbf{H}_n} (L_{n+1} - 1)^j \, y \mid b \in B, \; y \in Y, \; \ell \leq j < r \right\}$$

and set $M_r := 0$. *Then the following holds:*

a) *For every* ℓ, M_ℓ *is an* $\mathbf{H}_{n+1}$*-module.*
b) *It is* $0 = M_r \lneq M_{r-1} \cdots \lneq M_1 \lneq M_0 = \mathrm{Ind}_n(D^\lambda)$.
c) *Set* $N_\ell := M_\ell / M_{\ell+1}$ *for* $0 \leq \ell \leq r-1$. *Then* $T_0 = L_1$ *acts trivially on* N_ℓ *and* $\mathrm{Res}_{\mathfrak{S}_{n+1}}(N_\ell)$ *is isomorphic to* $\widehat{\mathrm{Ind}}_n \left(D^{\lambda^{(r)}} \right)$, *where we set* $\widehat{\mathrm{Ind}}_n := \mathrm{Ind}_{K[\mathfrak{S}_n]}^{K[\mathfrak{S}_{n+1}]}$.

Corollary 2.18. *Let* D^λ *be an irreducible* $\mathbf{H}_n$*-module for a multipartition* $\lambda = (\emptyset, \ldots, \emptyset, \lambda^{(r)}) \vdash_r n$. *Suppose* $t \in \mathbb{N}$ *is the number of irreducible constituents of the* $K[\mathfrak{S}_{n+1}]$*-module* $\widehat{\mathrm{Ind}}_n(D^{\lambda^{(r)}})$. *Then the number of irreducible constituents of* $\mathrm{Ind}_n(D^\lambda)$ *is exactly the product* rt.

Proposition 2.19. *Suppose* $q = 1$. *Let* $0 \neq M$ *be an* $\mathbf{H}_n$*-module. Then* $\mathrm{Ind}_n(M)$ *has at least* $2r$ *irreducible constituents.*

Proof. As induction is exact it suffices to prove the statement for $M = D^\lambda \neq 0$ for a multipartition $\lambda = (\emptyset, \ldots, \emptyset, \lambda^{(r)}) \vdash_r n$.
The induced module $\widehat{\mathrm{Ind}}_n(D^{\lambda^{(r)}})$ has at least 2 irreducible constituents by [Sch17, Theorem 1.1]. The claim now follows from Corollary 2.18. □

We finish this subsection by describing the socle of the induced modules, as this can be obtained with barely any additional work and complements the branching rules for $q \neq 1$, cf. [Ari06, Vaz02].

Proposition 2.20. *Assume the setting and notation of Proposition 2.17. Then the socle of* $\mathrm{Ind}_n(D^\lambda)$ *is contained in* M_{r-1}. *More precisely, the socle of* $\mathrm{Ind}_n(D^\lambda)$ *is isomorphic to the socle of* $\widehat{\mathrm{Ind}}_n(D^{\lambda^{(r)}})$ *where* T_0 *acts trivially.*

Proof. Clearly, the L_i act trivially on the socle of an $\mathbf{H}_{n+1}$-module.
It is easily checked that the common eigenspace of the L_i with respect to the eigenvalue 1 on $\mathrm{Ind}_n(D^\lambda)$ is exactly M_{r-1}. By Proposition 2.17 the restriction $\mathrm{Res}_{\mathfrak{S}_{n+1}}(M_{r-1})$ is isomorphic to $\widehat{\mathrm{Ind}}_n(D^{\lambda^{(r)}})$, yielding the claim.

□

Remark 2.21. *The socle of* $\widehat{\mathrm{Ind}}_n(D^{\lambda^{(r)}})$ *has been studied extensively by Kleshchev in his groundbreaking series of papers in the early 90's, cf.* [Kle05] *for a survey. In particular, he defines refined induction and restriction functors and shows that these can be defined in terms of adding and removing certain nodes. As this, too, yields a crystal, analogous to the ones defined for* $2 \leq e \leq \infty$, *we could also have used Kleshchev's results instead of* [Sch17, Theorem 1.1] *to show that* $\widehat{\mathrm{Ind}}_n(D^{\lambda^{(r)}})$ *has at least* 2 *constituents.*

2.3.3. *Main Theorem.* We drop our conditions to obtain a result on arbitrary Ariki-Koike algebras with invertible parameters:

Theorem 2.22. *Let t be the number of* $\sim_q$*-equivalence classes on* $(Q_1, \ldots, Q_r)$. *Then for any* $\mathbf{H}_n$*-module* $M \neq 0$ *the number of constituents of* $\mathrm{Ind}_n(M)$ *is at least* $r + t$. *In particular,* $\mathrm{Ind}_n(M)$ *is reducible.*

Proof. Reorder the Q_i such that $\mathbf{Q} := (Q_1, \dots, Q_r) = \mathbf{Q}_1 \coprod \cdots \coprod \mathbf{Q}_t$ is a concatenation of q-connected sequences that are pairwise not q-connected. Let $1 \leq j \leq t$. By Propositions 2.19, 2.13, and 2.11 and the definition of $\mathrm{Ind}_{n,t}$ we see that $\mathrm{Ind}_{n,t}(F_n(M))$ has at least $\sum_{j=1}^{t}(|\mathbf{Q}_j| + 1)$ irreducible constituents, where F_n is the natural equivalence from Theorem 2.2. By Corollary 2.8 we conclude that $\mathrm{Ind}_n(M)$ has at least $\sum_{j=1}^{t}(|\mathbf{Q}_j| + 1) = r + t$ constituents. □

Remark 2.23. *We call any subalgebra $\mathbf{H}'_n$ of $\mathbf{H}_n$ generated by a subset of the generators $T_0, \dots, T_{n-1}$ a* parabolic subalgebra of $\mathbf{H}_n$. *Adapting the proof of* [Sch17, Thm 1.1] *by using the Mackey formula from* [KMW18] *one can show that* $\mathrm{Ind}_{\mathbf{H}'_n}^{\mathbf{H}_n}(M)$ *is reducible for any non-zero $\mathbf{H}'_n$-module M, unless $\mathbf{H}'_n = \mathbf{H}_n$. However, if $\mathbf{H}'_n = \mathbf{H}_{n-1}$, then the statement in Theorem 2.22 is much stronger in general.*

2.4. **Degenerate cyclotomic Hecke algebras.** As already mentioned the Ariki-Koike algebras at $q = 1$ are generally not isomorphic to the so-called degenerate cyclotomic Hecke algebras. However, a result analogous to Theorem 2.22 still holds:
For a non-negative integer n denote by $\mathbf{h}_n$ the *degenerate affine Hecke algebra over K* as defined by Drinfel'd, cf. [Dri86], i.e. as a vector space it is $\mathbf{h}_n \cong K[x_1, \dots, x_n] \otimes K[\mathfrak{S}_n]$, the tensor product of the polynomial ring over K in n variables $x_1, \dots, x_n$ and the group algebra over K of the symmetric group $\mathfrak{S}_n$. Multiplication is defined such that $K[x_1, \dots, x_n] \otimes 1$ and $1 \otimes K[\mathfrak{S}_n]$ are both subalgebras and additionally we have

$$s_i x_j = x_j s_i \text{ if } j \neq i, i+1, \quad s_i x_{i+1} = x_i s_i + 1, \quad x_{i+1} s_i = s_i x_i + 1,$$

for all sensible values for i and j, where $s_i = (i, i+1)$ is the i'th standard Coxeter generator of $\mathfrak{S}_n$. Now let $r \geq 1$ and $\mathbf{s} = (s_1, \dots, s_r)$ in $\widetilde{\mathbb{Z}^r_{\geq 0}}$. Then the *degenerate cyclotomic Hecke algebra* $\mathbf{h}_n^{\mathbf{s}}$ is defined as the quotient

$$\mathbf{h}_n^{\mathbf{s}} := \mathbf{h}_n \,/\, \langle (x_1 - s_1) \cdots (x_1 - s_r) \rangle \,.$$

The algebra $\mathbf{h}_n^{\mathbf{s}}$ embeds into $\mathbf{h}_{n+1}^{\mathbf{s}}$ and the corresponding induction functor is exact.

Following Kleshchev (cf. [Kle05]) we can again define refined functors i-Ind for $0 \leq i \leq e := \mathrm{char}(K)$ and then repeat what we did for $e = \infty$. An analogue of Proposition 2.12 holds for $\mathbf{h}_n$. In particular, we once again obtain a crystal graph isomorphism to $B_e(\mathbf{s})$, cf. [Kle05, 10.3.5] and as in Proposition 2.13 we obtain the following:

Theorem 2.24. *Let $0 \neq M \in \mathbf{h}_n^{\mathbf{s}}$. Then the induced module $\mathrm{Ind}_{\mathbf{h}_n^{\mathbf{s}}}^{\mathbf{h}_{n+1}^{\mathbf{s}}}(M)$ has at least $r + 1$ constituents. In particular, it is reducible.*

Remark 2.25. *If K has characteristic* 0*, then by* [BK09, Cor 2] *the degenerate algebra $\mathbf{h}_n^{\mathbf{s}}$ is isomorphic to the Ariki-Koike algebra $\mathbf{H}_{n,r}(X; X^{s_1}, \dots, X^{s_r})$ over $K(X)$, where X is an indeterminate. Hence, in characteristic zero Theorem 2.24 already follows from Proposition 2.13.*

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