Blocks of unipotent characters in the finite general linear group

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We take q a power of a prime p, and $n \geq 1$ an integer. Let $G = GL(n,q)$ be the set of n by n non-singular matrices with coefficients in a finite field $\mathbf{F}_q$ with q elements.

1) Characters, blocks

The irreducible characters of G have been described by J. A. Green (1955), and then parametrized again by the Deligne-Lusztig Theory. Among them are the *unipotent* characters, which are parametrized by the partitions of n.

The r-blocks of G ($r \neq p$ an odd prime) have been described by P. Fong and B. Srinivasan (1982).

Theorem 1. *(Fong-Srinivasan) Let $r \neq p$ be an odd prime. Then two unipotent characters belong to the same r-block of G if and only if the partitions labeling them have the same e-core, where e is the multiplicative order of q modulo r.*

We would like to give a construction of the "unipotent blocks" which doesn't involve any prime. To do that, we adapt the objects and methods used in the symmetric group by B. Külshammer, J. B. Olsson and G. R. Robinson (*Generalized blocks for symmetric groups*, Invent. Math. **151**(3), 513-552 (2003)).

As can be seen from Theorem 1, this involves the combinatorics behind partitions of n.

Take $e = 3$ and the partition $\lambda = (4, 4, 2, 1)$ of $n = 11$.

Young diagram of λ:

Hooks, e-*hooks*:

7	5	3	2
6	4	2	1
3	1		
1			

We remove one after the other the e-hooks:

5	4	3	2
4	3	2	1

4	3	2	1
1			

2
1

e-*core* of λ:

2) Conjugacy classes

Let $g \in G$. Then g is conjugate in G to its rational canonical form (which is essentially unique)

$$\mathcal{U}(g) = \begin{pmatrix} \mathcal{U}_g(f_1) & & \\ & \ddots & \\ & & \mathcal{U}_g(f_s) \end{pmatrix},$$

where $f_1, \ldots, f_s \in \mathbf{F}_q[X]$ are the irreducible divisors of the minimal polynomial $Min(g)$ of g over $\mathbf{F}_q$, and $\mathcal{U}_g(f_i)$ has a block structure with companion matrices of f_i on the diagonal and possibly some identity matrices on the upper-diagonal. Two elements of G are conjugate if and only if they have the same rational canonical form.

3) Sections

Let $d \geq 1$ be an integer. We separate in $\mathcal{U}(g)$ the blocks corresponding to polynomials of degree divisible by d from the others.
An element g of G is *d-regular* if none of the irreducible divisors of $Min(g)$ distinct from $X-1$ has degree divisible by d.
Any element of G can be written uniquely as the commuting product of a d-regular element and a d-element.
The idea behind these definitions is that they are "compatible" with the Murnaghan-Nakayama Rule for unipotent characters. In the symmetric group, we can isolate cycles. In $GL(n, q)$, we can isolate polynomials.

We let $\mathcal{C}$ be the set of d-regular elements of G.

4) d-blocks

Two characters $\chi, \psi \in$Irr(G) are said to be *directly $\mathcal{C}$-linked* if

$$< \chi, \psi >_{\mathcal{C}} := \frac{1}{|G|} \sum_{g \in \mathcal{C}} \chi(g)\overline{\psi(g)} \neq 0.$$

The transitive closure of direct $\mathcal{C}$-linking gives a partition of Irr(G) into *d-blocks*. Similarly, we define d-blocks of unipotent characters.

Theorem 2. *(G.) If two unipotent characters χ_λ and χ_μ belong to the same unipotent d-block of G, then λ and μ have the same d-core.*

The proof is by induction on n, using the Murnaghan-Nakayama Rule for unipotent characters.

This is one direction of an analogue of the Nakayama Conjecture.

Theorem 3. *(G.) Let $n = 2$ or 3 and $G = GL(n, q)$. Then, for any $1 \leq d \leq n$, the unipotent d-blocks of G satisfy an analogue of the Nakayama Conjecture.*

The special case when $d = 1$ is easily dealt with. The 1-regular elements of $GL(n, q)$ are precisely the unipotent elements.

Theorem 4. *(Geck, G.) The unipotent 1-blocks and the 1-blocks of $GL(n, q)$ satisfy an analogue of the Nakayama Conjecture.*

In fact, each irreducible character is directly linked across unipotent elements to the trivial character.

For $d > 1$, this is work in progress!!